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EXPONENTIAL STABILITY AND SPECTRAL ANALYSIS OF THE INVERTED PENDULUM SYSTEM UNDER TWO DELAYED POSITION FEEDBACKS

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Abstract. In this paper, we examine the stability of a linearized inverted pendulum system with two delayed position feedbacks. The semigroup approach is adopted in investigation for the well-posedness of the closed loop system. We prove that the spectrum of the system is located in the left complex half-plane and its real part tends to $-\infty$ when the feedback gains satisfy some additional conditions. The asymptotic eigenvalues of the system is presented. By estimating the norm of the Riesz spectrum projection of the system operator that does not have the uniformly upper bound, we show that the eigenfunctions of the system do not form a basis in the state Hilbert space. Furthermore, the spectrum determined growth condition of the system is concluded and the exponential stability of the system is then established. Finally, numerical simulation is presented by applying the MATLAB software.

1. Introduction

It is well known that time-delay often appears in many biological, robotic, and electrical systems and mechanical applications [6, 16, 17], and it has attracted many engineers and mathematicians to work on the system with delays [1,2,7,10,13]. For a detailed survey, we refer the reader to [13] and the references therein. On the other hand, intentionally introduced time delay has been used successfully in feedback control design [6,20] and filter design [15]. A control design called "proportional minus delay controller" (PMD) was introduced in [17, 18] with the objective of improving the performance of the system.

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In general, a simple position feedback does not stabilize a controlled system and some additional feedbacks such as the velocity etc are needed to force the system [1, 16]. In the absence of measurement of the velocity, an observer is always adopted to construct the state but this might degrade the performance to some extent [8]. However, in applications, some slightly damped or undamped mechanical systems, such as a rotary crane and a robotic manipulator, their free vibrations always decay very slowly [10]. So, it is natural to use the time-delay as a control force to the system and the stability issue of systems with delay inputs is, therefore, of theoretical and practical importance.

One of the simplest problems in robotics is that of controlling the position of a single-link rotational joint using a motor placed at the pivot, mathematically that is an inverted pendulum to which one can apply a torque as an external force [16]. The motion of the inverted pendulum with an external torque can be formulated as the following second-order linearized differential equation

$$\ddot{y}(t) - \frac{g}{\ell} y(t) = u(t), \tag{1.1}$$

where y denotes the angular displacement from the inverted equilibrium, g denotes the acceleration due to gravity, ℓ is the length of the pendulum, and $u(t)$ denotes the value of the external torque at time t. We use the two delayed positions as the control input to (1.1):

$$u(t) = \hat{a} y(t-\tau) + \hat{b} y(t-2\tau).$$

Then (1.1) becomes

$$\ddot{y}(t) - \frac{g}{\ell} y(t) = \hat{a} y(t-\tau) + \hat{b} y(t-2\tau). \tag{1.2}$$

Rescaling the time $t \to t/\tau$ normalizes the delay τ to 1, we obtain the following closed-loop system:

$$\ddot{y}(t) + k y(t) = a y(t-1) + b y(t-2), \tag{1.3}$$

where $k = -g\tau^2/\ell < 0$ is the physical parameter and $a = \hat{a}\tau^2$ and $b = \hat{b}\tau^2$ are two feedback gains.

When $k < 0$, Atay in [1] gave a necessary and sufficient condition for a and b such that the roots of characteristic determinant of system (1.3) lie in the left complex half-plane and we restate here as Lemma 3.1. In this paper, we use the semigroup theory to explain the well-posedness of (1.3) and give a rigorous spectral analysis to obtain the spectrum properties and the asymptotic expressions for eigenvalues. Moreover, by estimating the norm of the Riesz spectrum projection of the system operator, we show that the generalized eigenfunctions of the system do not form a Riesz basis in Hilbert space. Finally, we also prove that the spectrum determined growth condition is held and therefore the exponential stability is established.

The paper is structured as follows. In Sec. 2, we formulate the closed-loop system into an abstract evolution equation and the C_0-semigroup approach is used to prove the well-posedness of the system. Section 3 is devoted to the detailed spectral analysis of the system. We obtain that with some conditions required on the feedback gains, all eigenvalues λ_n are located in the left complex half-plane and their real parts $\operatorname{Re}\lambda_n$ tend to $-\infty$ as $n \to \infty$. The asymptotic spectral expression is also presented. In Sec. 4, we show that the generalized eigenfunctions of the system do not form a Riesz basis for the Hilbert state space. In Sec. 5, the spectrum determined growth condition is held and the exponential stability of the system is then established. Finally, some numerical simulations are presented in Sec. 6 to illustrate the eigenvalue distributions and the stability of the system.

2. Setup and well-posedness of the system

In this section, we transform system (1.3) into an abstract evolution equation and then discuss the well-posedness of the system. Denote

$$Z(t) = (z_1(t), z_2(t))^T,$$

where $z_1(t) = y(t), z_2(t) = \dot{z}_1(t)$, and ^{T} denotes the transpose of a vector or a matrix. Then (1.3) can be rewritten as follows:

$$\dot{Z}(t) = A_0 Z(t) + A_1 Z(t-1) + A_2 Z(t-2), \tag{2.1}$$

where

$$A_0 = \begin{pmatrix} 0 & 1 \\ -k & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}, \tag{2.2}$$

$k < 0$ is the physical parameter given by (1.3), and $a \neq 0$ and $b \neq 0$ are two feedback gains. It is natural to set the initial date of (2.1) as follows:

$$\begin{cases} Z(0) = Z_0 = (z_{10}, z_{20})^T, \\ Z(s) = \Phi(s), \quad s \in [-2, 0], \end{cases} \tag{2.3}$$

where $Z_0 \in \mathbb{C}^2$ and $\Phi \in L^2([-2,0], \mathbb{C}^2)$. We consider system (1.3) in the Hilbert state space

$$\mathcal{H} = \mathbb{C}^2 \times L^2([-2,0], \mathbb{C}^2)$$

equipped with the usual inner product:

$$\langle X, Y \rangle_{\mathcal{H}} = \langle x, y \rangle_{\mathbb{C}^2} + \int_{-2}^{0} \langle f(s), g(s) \rangle_{\mathbb{C}^2}\, ds, \tag{2.4}$$

where $X = (x, f)^T \in \mathcal{H}$ and $Y = (y, g)^T \in \mathcal{H}$. Define a linear operator $\mathcal{A} : \mathcal{H} \to \mathcal{H}$ as follows:

$$\mathcal{A} \begin{pmatrix} x \\ f \end{pmatrix} = \begin{pmatrix} A_0 & A_1\delta_1 + A_2\delta_2 \\ 0 & \dfrac{d}{ds} \end{pmatrix} \begin{pmatrix} x \\ f \end{pmatrix} \tag{2.5}$$

with

$$D(\mathcal{A}) = \Big\{(x,f)^T \in \mathcal{H} \;\Big|\; f \in H^1([-2,0],\mathbb{C}^2),\ f(0) = x\Big\}, \tag{2.6}$$

where

$$\delta_1 f = f(-1), \quad \delta_2 f = f(-2) \quad \forall f \in C[-2,0]$$

and A_0 and A_1 are given by (2.2).

Denote

$$\begin{cases} f(t,s) = Z(t+s), & s \in [-2,0], \\ X(t) = (Z(t), f(t,s))^T, & \\ X(0) = X_0 := (Z_0, \Phi(s))^T, & s \in [-2,0]; \end{cases} \tag{2.7}$$

then system (2.1) with (2.3) can be formulated into the following abstract evolution equation on $\mathcal{H}$:

$$\begin{cases} \dfrac{dX(t)}{dt} = \mathcal{A}X(t), t > 0, \\ X(0) = X_0. \end{cases} \tag{2.8}$$

Now we have the following two lemmas on the properties of $\mathcal{A}$.

Lemma 2.1. *Let $\mathcal{A}$ be given by* (2.5) *and* (2.6) *and let*

$$\langle X, Y\rangle_1 = \langle x, y\rangle_{\mathbb{C}^2} + \int_{-1}^{0} q_1(s)\,\langle f(s), g(s)\rangle_{\mathbb{C}^2}\, ds + \int_{-2}^{-1} q_2(s)\,\langle f(s), g(s)\rangle_{\mathbb{C}^2}\, ds, \tag{2.9}$$

where $X = (x,f)^T \in \mathcal{H}$, $Y = (y,g)^T \in \mathcal{H}$, and

$$\begin{cases} q_1(s) = a^2 s^2 + b^2 & \forall s \in [-1,0], \\ q_2(s) = -b^2 s & \forall s \in [-2,-1] \end{cases} \tag{2.10}$$

are two positive and bounded functions. Then $\langle \cdot,\cdot\rangle_1$ is an inner product in $\mathcal{H}$ whose induced norm is equivalent to the general one induced by (2.4). *Moreover, there is a positive constant $M > 0$ such that*

$$\mathrm{Re}\langle \mathcal{A}X, X\rangle_1 \le M\langle X, X\rangle_1 \quad \forall X \in D(\mathcal{A}). \tag{2.11}$$

Hence, $\mathcal{A} - M$ is dissipative in $\mathcal{H}$.

Proof. The first conclusion is obvious and we only need to show (2.11). For each $X = (x, f(s))^T \in D(\mathcal{A})$, we have

$$\langle \mathcal{A}X, X\rangle_1 = \langle A_0 x + A_1 f(-1) + A_2 f(-2), x\rangle_{\mathbb{C}^2} + \int_{-1}^{0} q_1(s)\left\langle \frac{d}{ds} f(s), f(s)\right\rangle_{\mathbb{C}^2} ds + \int_{-2}^{-1} q_2(s)\left\langle \frac{d}{ds} f(s), f(s)\right\rangle_{\mathbb{C}^2} ds.$$

A direct computation yields

$$
\begin{aligned}
&\mathrm{Re}\langle \mathcal{A}X, X\rangle_1 \\
&\le \|A_0\|\|x\|_{\mathbb{C}^2}^2 + \|A_1\|\|f(-1)\|_{\mathbb{C}^2}\|x\|_{\mathbb{C}^2} + \|A_2\|\|f(-2)\|_{\mathbb{C}^2}\|x\|_{\mathbb{C}^2} \\
&\quad + \frac{1}{2}\int_{-1}^{0} q_1(s)\frac{d}{ds}\|f(s)\|_{\mathbb{C}^2}^2 ds + \frac{1}{2}\int_{-2}^{-1} q_2(s)\frac{d}{ds}\|f(s)\|_{\mathbb{C}^2}^2 ds \\
&\le \|A_0\|\|x\|_{\mathbb{C}^2}^2 + \frac{1}{2}\left(\|A_1\|^2\|f(-1)\|_{\mathbb{C}^2}^2 + \|x\|_{\mathbb{C}^2}^2\right) \\
&\quad + \frac{1}{2}\left(\|A_2\|^2\|f(-2)\|_{\mathbb{C}^2}^2 + \|x\|_{\mathbb{C}^2}^2\right) \\
&\quad + \frac{1}{2}\left(q_1(s)\|f(s)\|_{\mathbb{C}^2}^2\Big|_{-1}^{0} - \int_{-1}^{0} q_1'(s)\|f(s)\|_{\mathbb{C}^2}^2 ds\right) \\
&\quad + \frac{1}{2}\left(q_2(s)\|f(s)\|_{\mathbb{C}^2}^2\Big|_{-2}^{-1} - \int_{-2}^{-1} q_2'(s)\|f(s)\|_{\mathbb{C}^2}^2 ds\right) \\
&= \left(\|A_0\| + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}q_1(0)\right)\|x\|_{\mathbb{C}^2}^2 \\
&\quad + \frac{1}{2}\|f(-1)\|_{\mathbb{C}^2}^2\left(\|A_1\|^2 - q_1(-1) + q_2(-1)\right) \\
&\quad + \frac{1}{2}\|f(-2)\|_{\mathbb{C}^2}^2\left(\|A_2\|^2 - q_2(-2)\right) + \int_{-1}^{0} \frac{-q_1'(s)}{2q_1(s)}\cdot q_1(s)\|f(s)\|_{\mathbb{C}^2}^2 ds \\
&\quad + \int_{-2}^{-1} \frac{-q_2'(s)}{2q_2(s)}\cdot q_2(s)\|f(s)\|_{\mathbb{C}^2}^2 ds.
\end{aligned}
$$

Note that $\|A_1\| = |a|$ and $\|A_2\| = |b|$, from (2.10), we have $q_1(0) = \|A_2\|^2$,

$$
\begin{cases}
\|A_1\|^2 - q_1(-1) + q_2(-1) = 0, \\
\|A_2\|^2 - q_2(-2) < 0, \quad q_i'(s) < 0,
\end{cases}
$$

and

$$
\frac{-q_1'(s)}{2q_1(s)} = \frac{-\|A_1\|^2 s}{\|A_1\|^2 s^2 + \|A_2\|^2} \le \frac{-\|A_1\|^2 s}{2\|A_1\||s|\cdot\|A_2\|} = \frac{\|A_1\|}{2\|A_2\|}
$$

for $s \in [-1, 0]$ and

$$
\frac{-q_2'(s)}{2q_2(s)} = \frac{\|A_2\|^2}{-2\|A_2\|^2 s} = \frac{1}{-2s} \le \frac{1}{2}
$$

for $s \in [-2,-1]$. Let

$$M = \max\left\{\|A_0\| + \frac{1}{2}\|A_2\|^2 + 1,\ \frac{\|A_1\|}{2\|A_2\|}\right\}.$$

Then we conclude

$$\begin{aligned}&\mathrm{Re}\langle \mathcal{A}X, X\rangle_1\\ &\quad \le M\left[\|x\|^2_{\mathbb{C}^2} + \int_{-1}^{0} q_1(s)\|f(s)\|^2_{\mathbb{C}^2}ds + \int_{-2}^{-1} q_2(s)\|f(s)\|^2_{\mathbb{C}^2}ds\right]\\ &\quad = M\langle X, X\rangle_1.\end{aligned}$$

This is the required (2.11). The proof is complete. □

Lemma 2.2. *Let $\mathcal{A}$ be given by* (2.5) *and* (2.6) *and let*

$$\Delta(\lambda) = \lambda - A_0 - A_1e^{-\lambda} - A_2e^{-2\lambda} = \begin{pmatrix} \lambda & -1 \\ k - ae^{-\lambda} - be^{-2\lambda} & \lambda \end{pmatrix}.$$

If $\det \Delta(\lambda) \neq 0$*, then* $\lambda \in \rho(\mathcal{A})$*, where* $\rho(\mathcal{A})$ *is the resolvent set of* $\mathcal{A}$*. Moreover, the resolvent* $(\lambda - \mathcal{A})^{-1}$ *of* $\mathcal{A}$ *is compact and we have the following expressions*:

$$(\lambda - \mathcal{A})^{-1}Y = X = (x, f)^T \in D(\mathcal{A}) \quad \forall Y = (y, g)^T \in \mathcal{H},$$

$$\begin{aligned} x &= \Delta(\lambda)^{-1}\left[y + A_1\int_{-1}^{0} e^{-\lambda(1+s)}g(s)ds + A_2\int_{-2}^{0} e^{-\lambda(2+s)}g(s)ds\right],\\ f(s) &= e^{\lambda s}x + \int_{s}^{0} e^{\lambda(s-r)}g(r)dr. \end{aligned} \tag{2.12}$$

In particular,

$$\sigma(\mathcal{A}) = \sigma_p(\mathcal{A}) = \{\lambda \in \mathbb{C} \mid \det \Delta(\lambda) = 0\}.$$

Furthermore, each $\lambda \in \sigma_p(\mathcal{A})$ *is geometrically simple and its eigenfunction has the following form*:

$$\phi_\lambda = (x, e^{\lambda s}x)^T, \tag{2.13}$$

where $x = (1, \lambda)^T$.

Proof. Let $\lambda \in \mathbb{C}$ and $Y = (y, g)^T \in \mathcal{H}$. By the resolvent equation

$$(\lambda - \mathcal{A})X = Y, \quad X = (x, f)^T \in D(\mathcal{A}),$$

we have

$$\begin{cases} \lambda x - A_0x - A_1f(-1) - A_2f(-2) = y, \\ \lambda f(s) - \dfrac{d}{ds}f(s) = g(s), \quad s \in [-2, 0], \\ f(0) = x. \end{cases} \tag{2.14}$$

From the last two equations of (2.14), we get a unique solution $f(s)$ by

$$f(s) = e^{\lambda s}x + \int_s^0 e^{\lambda(s-r)} g(r)dr. \tag{2.15}$$

Substituting this into the first equation of (2.14), we have

$$\lambda x - A_0 x - A_1 \left(e^{-\lambda} x + \int_{-1}^0 e^{\lambda(-1-r)} g(r)dr \right) - A_2 \left(e^{-2\lambda} x + \int_{-2}^0 e^{\lambda(-2-r)} g(r)dr \right) = y,$$

i.e.,

$$\lambda x - A_0 x - A_1 e^{-\lambda} x - A_2 e^{-2\lambda} x = y + A_1 \int_{-1}^0 e^{\lambda(-1-r)} g(r)dr + A_2 \int_{-2}^0 e^{\lambda(-2-r)} g(r)dr.$$

Thus, we have

$$\Delta(\lambda)x = y + A_1 \int_{-1}^0 e^{\lambda(-1-r)} g(r)dr + A_2 \int_{-2}^0 e^{\lambda(-2-r)} g(r)dr.$$

If $\det \Delta(\lambda) \neq 0$, then

$$x = \Delta(\lambda)^{-1} \left[y + A_1 \int_{-1}^0 e^{\lambda(-1-r)} g(r)dr + A_2 \int_{-2}^0 e^{\lambda(-2-r)} g(r)dr \right], \tag{2.16}$$

and $(x, f)^T \in D(\mathcal{A})$ is uniquely determined. Hence $\lambda \in \rho(\mathcal{A})$, $\mathcal{A}$ is closed in $\mathcal{H}$, and $(\lambda - \mathcal{A})^{-1}$ is compact. Finally, if $\det \Delta(\lambda) = 0$ for $\lambda \in \mathbb{C}$, the equation $\Delta(\lambda)x = 0$ will have a nontrivial solution $x = (1, \lambda)^T \in \mathbb{C}^2$. Obviously, $(x, e^{\lambda s}x)^T \in D(\mathcal{A})$, and

$$(\lambda - \mathcal{A}) \begin{pmatrix} x \\ e^{\lambda s} x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Hence $\lambda \in \sigma_p(\mathcal{A})$ and

$$\sigma(\mathcal{A}) = \sigma_p(\mathcal{A}) = \{\lambda \in \mathbb{C} \mid \det \Delta(\lambda) = 0\}.$$

The proof is complete. □

By Lemmas 2.1 and 2.2, we have the well-posedness of system (2.8) as the following theorem.

Theorem 2.1. *Let $\mathcal{A}$ be given by* (2.5) *and* (2.6)*. Then $\mathcal{A}$ generates a C_0-semigroup $e^{\mathcal{A}t}$ in $\mathcal{H}$.*

Proof. Note that from Lemma 2.1 we have that $\mathcal{A} - M$ is dissipative in $\mathcal{H}$ and by Lemma 2.2 we get that the right complex half-plane belongs to the resolvent set of $\mathcal{A} - M$. Then, by the Lumer–Phillips theorem, $\mathcal{A} - M$ generates a C_0-semigroup of contractions $e^{(\mathcal{A}-M)t}$ in $\mathcal{H}$. Moreover, the bounded perturbation theorem of C_0-semigroups implies that $\mathcal{A}$ generates a C_0-semigroup $e^{\mathcal{A}t}$ in $\mathcal{H}$ (see [12]). The proof is complete. □

3. Spectral analysis of the system

In this section, we analyze the spectrum distribution of the system operator $\mathcal{A}$. Some analytic methods in [4] will be adopted here. From Lemma 2.2, we have that $\lambda \in \sigma(\mathcal{A})$ if and only if $\det \Delta(\lambda) = 0$. So we only need to discuss the roots of $\det \Delta(\lambda)$. Note that

$$\begin{aligned}\det \Delta(\lambda) &= \det(\lambda - A_0 - A_1 e^{-\lambda} - A_2 e^{-2\lambda}) \\ &= \begin{vmatrix} \lambda & -1 \\ k - ae^{-\lambda} - be^{-2\lambda} & \lambda \end{vmatrix} = \lambda^2 + k - ae^{-\lambda} - be^{-2\lambda}.\end{aligned}$$

We have the following lemma directly (see Proposition 2 of [1]).

Lemma 3.1 (see [1, Proposition 2])**.** *Let $k < 0$ and let λ be a root of $\det \Delta(\lambda)$, that is, $\lambda \in \sigma(\mathcal{A})$. Then λ has a negative real part,* $\operatorname{Re} \lambda < 0$, *if and only if*

$$k > -1, \quad -k < b < \left(\frac{\pi}{2}\right)^2 - k, \quad -2b \cos\sqrt{b+k} < a < k - b. \tag{3.1}$$

Moreover, the stability region of roots with positive real parts is shown in Fig. 1*, where the boldface number in each region is the number of roots with positive real parts.*

Now we analyze the spectrum of the system operator $\mathcal{A}$. From now on, for brevity, we use the notation

$$h(\lambda) = \det \Delta(\lambda) = \lambda^2 + k - ae^{-\lambda} - be^{-2\lambda}. \tag{3.2}$$

Lemma 3.2. *Let $h(\lambda)$ be given by* (3.2)*. Then each root of $h(\lambda)$ is simple except at most four roots and these nonsimple roots satisfy the following polynomial equation*:

$$4b\lambda^4 + 8b\lambda^3 + \left[8bk + 4b + a^2\right]\lambda^2 + \left[8bk + 2a^2\right]\lambda + 4bk^2 + a^2 k = 0. \tag{3.3}$$

Proof. Let $h(\lambda)$ be given by (3.2). Then we have

$$h'(\lambda) = 2\lambda + ae^{-\lambda} + 2be^{-2\lambda}. \tag{3.4}$$

If λ is a root of $h(\lambda)$ with multiplicity at least two, then we have

$$h(\lambda) = h'(\lambda) = 0.$$

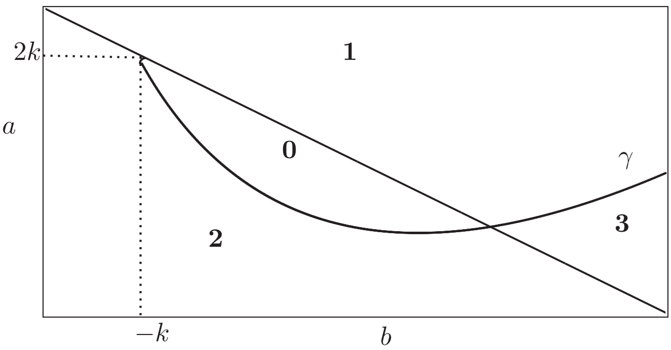

Fig. 1. The stability region for (3.1) in $b-a$ plane. γ is the curve generated by $a=-2b\cos\sqrt{b+k}$ and the boldface number in each region is the number of roots with positive real parts.

So, by (3.2) and (3.4), we obtain

$$\begin{cases} ae^{-\lambda} = 2\left[\lambda^2+\lambda+k\right], \\ be^{-2\lambda} = -\left[\lambda^2+2\lambda+k\right]. \end{cases} \tag{3.5}$$

Note that

$$e^{-2\lambda} = (e^{-\lambda})^2.$$

So λ is a root of $h(\lambda)$ with multiplicity at least two if and only if λ satisfies the following equation:

$$4b\left[\lambda^2+\lambda+k\right]^2 = -a^2\left[\lambda^2+2\lambda+k\right].$$

This yields (3.3), and there are only at most four roots such that their multiplicities at least two. The proof is complete. □

Lemma 3.3. *Let $k<0$, let $h(\lambda)$ with $\lambda\in\mathbb{C}$ be given by* (3.2)*, and let condition* (3.1) *hold. Then there is at most four real roots of $h(\lambda)$ and each real root is negative if exists.*

Proof. Let $\lambda = d$ with $d\in\mathbb{R}$. Then (3.2) implies that

$$h(d) = d^2+k-ae^{-d}-be^{-2d}.$$

Since condition (3.1) holds, we have by Lemma 3.1 $-1<k<0$, $b>0$, $a<0$, and hence each real root of $h(\lambda)$ is negative if exists. Let

$$f(d) = e^{2d}h(d) = d^2e^{2d}+ke^{2d}-ae^d-b.$$

Then we have

$$f'(d) = 2(d^2 + d + k)e^{2d} - ae^d.$$

Let

$$g(d) = e^{-d} f'(d) = 2(d^2 + d + k)e^d - a.$$

Then $g(d)$ and $f'(d)$ have the same sign. Note that

$$g'(d) = 2e^d(d^2 + 3d + k + 1);$$

this implies that $g'(d)$ have two negative roots:

$$d_i = \frac{-3 \pm \sqrt{5 - 4k}}{2}, \quad i = 1, 2.$$

Hence $g(d)$ and $f'(d)$ have at most three real roots, and therefore $h(d)$ and $f(d)$ have at most four real roots and each real root is negative if existed. The proof is complete. □

Lemma 3.4. *Let $k < 0$, let $h(\lambda)$ with $\lambda \in \mathbb{C}$ be given by* (3.2)*, and let condition* (3.1) *hold. Then $h(\lambda)$ has infinitely many roots λ_n, $n \in \mathbb{N}$ in $\mathbb{C}^-$. Moreover, these roots satisfy the condition*

$$\operatorname{Re} \lambda_n \to -\infty \qquad \text{as } n \to \infty. \tag{3.6}$$

Proof. Note that $h(\lambda)$ is an entire function in λ. Then there are infinitely many roots in the complex plane. Moreover, by Lemma 3.1, these roots are located in the left complex half-plane. Furthermore, if $|\lambda|$ is sufficiently large and $\operatorname{Re} \lambda$ is bounded, then we have

$$|h(\lambda)| \geq |\lambda|^2 - 1 - |a| e^{-\operatorname{Re} \lambda} - b e^{-2 \operatorname{Re} \lambda} > 0.$$

This yields that $\operatorname{Re} \lambda_n \to -\infty$ as $n \to \infty$. The proof is complete. □

Now we are in a position to consider the asymptotic distribution of the roots of $h(\lambda)$. Let

$$f(\lambda) = e^{2\lambda} h(\lambda) = \lambda^2 e^{2\lambda} + k e^{2\lambda} - a e^{\lambda} - b. \tag{3.7}$$

When $\operatorname{Re} \lambda \to -\infty$, $f(\lambda)$ has the following asymptotic expression:

$$f(\lambda) = e^{2\lambda} h(\lambda) = \lambda^2 e^{2\lambda} - b + \mathcal{O}(e^{\lambda}). \tag{3.8}$$

By the Rouché theorem, in order to consider the asymptotic distribution of the roots of $f(\lambda)$, we only need to consider the following function:

$$\tilde{f}(\lambda) = \lambda^2 e^{2\lambda} - b, \tag{3.9}$$

which can be decomposed as follows:

$$\tilde{f}(\lambda) = (\lambda e^{\lambda} - \sqrt{b})(\lambda e^{\lambda} + \sqrt{b}), \quad b > 0. \tag{3.10}$$

Let

$$f_1(\lambda) = \lambda e^{\lambda} - \sqrt{b}, \quad f_2(\lambda) = \lambda e^{\lambda} + \sqrt{b}, \quad b > 0. \tag{3.11}$$

Now we consider the asymptotic roots of $f_i(\lambda)$, $i = 1, 2$, separately.

Proposition 3.1. *Let* $b > 0$ *and let* $f_1(\lambda)$ *be given by* (3.11). *Then*

$$f_1(\lambda) = \lambda e^{\lambda} - \sqrt{b}$$

has roots

$$\sigma(f_1(\lambda)) = \{\xi_n,\ \overline{\xi_n}\}_{n\in\mathbb{N}} \cup \{\nu_1\}, \tag{3.12}$$

where ν_1 *is the unique positive real root of* $f_1(\lambda)$ *and* ξ_n *has the following asymptotic expression*:

$$\xi_n = \left[\ln\sqrt{b} - \ln\left[\left(2n - \frac{1}{2}\right)\pi\right]\right] + i\left[\left(2n - \frac{1}{2}\right)\pi - \frac{\ln\left(2n - \frac{1}{2}\right)\pi}{\left(2n - \frac{1}{2}\right)\pi}\right] + \mathcal{O}(n^{-1}). \tag{3.13}$$

Proof. First, we find the real root of $f_1(\lambda)$. Let $\nu \in \mathbb{R}$ be a real root of $f_1(\lambda)$. Then it follows from $b > 0$ and $f_1(\nu) = 0$ that $\nu > 0$, i.e., there is only possible to have the positive real root of $f_1(\lambda)$. Due to the fact that

$$f_1'(\lambda) = e^{\lambda} + \lambda e^{\lambda} > 0 \quad (\text{when } \lambda > 0)$$

and

$$\lim_{\lambda\to 0} f_1(\lambda) = -\sqrt{b} < 0, \qquad \lim_{\lambda\to+\infty} f_1(\lambda) = +\infty > 0,$$

we see that $f_1(\lambda)$ has only one positive real root ν_1.

Next, since the complex roots of $f_1(\lambda)$ are symmetric with respect to the real axis, they will have the form (3.12), and hence the proof will be accomplished if we can show that ξ_n has the asymptotic expression (3.13).

Let $\xi = x + iy$ with $y > 0$ be a root of $f_1(\lambda)$. Then it follows from $f_1(\xi) = 0$ that

$$(x + iy)e^{x}(\cos y + i\sin y) = \sqrt{b},$$

which yields

$$e^{x}(x\cos y - y\sin y) = \sqrt{b} \tag{3.14}$$

and

$$e^{x}(x\sin y + y\cos y) = 0. \tag{3.15}$$

A direct computation from (3.15) yields

$$x = -\frac{y\cos y}{\sin y}. \tag{3.16}$$

Substituting this into (3.14) we have

$$e^{x} = -\frac{\sqrt{b}\sin y}{y}. \tag{3.17}$$

Due to the fact that $\sqrt{b} > 0$, $y > 0$, and $e^x > 0$, we have $\sin y < 0$, and hence we obtain

$$y \in ((2n-1)\pi,\ 2n\pi), \quad n \in \mathbb{N}. \tag{3.18}$$

Moreover, it follows from (3.17) that

$$x = \left[\ln\left(-\sqrt{b}\sin y\right) - \ln y\right]. \tag{3.19}$$

Substituting this into (3.16) we have

$$\ln\left(-\sqrt{b}\sin y\right) - \ln y + \frac{y\cos y}{\sin y} = 0.$$

Let

$$g(y) = \ln\left(-\sqrt{b}\sin y\right) - \ln y + \frac{y\cos y}{\sin y}.$$

Then we have

$$g'(y) = \frac{y\sin 2y - \sin^2 y - y^2}{y\sin^2 y} < 0,$$

where we have used (3.18). On the other hand,

$$\lim_{y\to(2n-1)\pi} g(y) = +\infty, \quad \lim_{y\to 2n\pi} g(y) = -\infty.$$

Hence there exists a unique root y_n, $n \in \mathbb{N}$, on each interval

$$((2n-1)\pi,\ 2n\pi), \quad n \in \mathbb{N},$$

such that $g(y_n) = 0$. For each $n \in \mathbb{N}$, by taking

$$x_n = \ln\frac{-\sqrt{b}\sin y_n}{y_n}, \tag{3.20}$$

we see that $\xi_n = x_n + iy_n$ is a root of $f_1(\lambda)$.

When $y_n > \sqrt{b}$, we have $x_n < 0$, and hence

$$y_n \to +\infty, \quad x_n \to -\infty \quad \text{as } n \to +\infty. \tag{3.21}$$

Moreover, by (3.16) and (3.17), we have respectively

$$\sin y_n = -\frac{y_n\cos y_n}{x_n}, \quad \sin y_n = -\frac{e^{x_n}y_n}{\sqrt{b}}.$$

This yields

$$x_n e^{x_n} = \sqrt{b}\cos y_n. \tag{3.22}$$

So, due to the fact that $x_n < 0$ and $\sqrt{b} > 0$, we have

$$\cos y_n < 0.$$

This and (3.18) yield

$$y_n \in \left((2n-1)\pi,\ (2n-\frac{1}{2})\pi\right), \quad n \in \mathbb{N}. \tag{3.23}$$

Furthermore, it follows from (3.21) and (3.22) that as $n \to +\infty$,

$$x_n e^{x_n} \to 0, \quad \cos y_n \to 0, \quad y_n - \left(2n - \frac{1}{2}\right)\pi \to 0.$$

Therefore, we obtain the form of y_n as follows:

$$y_n = \left(2n - \frac{1}{2}\right)\pi + \varepsilon_n, \quad \varepsilon_n \in \left(-\frac{\pi}{2}, 0\right), \tag{3.24}$$

where $\varepsilon_n \to 0$ as $n \to +\infty$. Substituting (3.24) in $g(y_n) = 0$ to obtain

$$0 = g(y_n) = \ln\sqrt{b} + \ln(-\sin y_n) - \ln y_n + \frac{y_n \cos y_n}{\sin y_n}.$$

This yields

$$\ln\sqrt{b} + \ln(\cos\varepsilon_n) - \ln y_n + \frac{y_n \sin\varepsilon_n}{-\cos\varepsilon_n} = 0$$

and

$$\sin\varepsilon_n = \frac{\cos\varepsilon_n}{y_n}\left[\ln\sqrt{b} + \ln(\cos\varepsilon_n) - \ln y_n\right].$$

Expanding by the Taylor series, we have

$$\sin\varepsilon_n = -\frac{\ln y_n}{y_n} + \mathcal{O}(n^{-1}) \quad \text{as } n \to +\infty.$$

Note that

$$\sin\varepsilon_n = \varepsilon_n - \frac{\varepsilon_n^3}{3!} + \cdots;$$

we have

$$\varepsilon_n = -\frac{\ln\left(2n - \frac{1}{2}\right)\pi}{\left(2n - \frac{1}{2}\right)\pi} + \mathcal{O}(n^{-1}).$$

Hence from (3.24) we immediately obtain the asymptotic expression of y_n:

$$y_n = \left(2n - \frac{1}{2}\right)\pi - \frac{\ln\left(2n - \frac{1}{2}\right)\pi}{\left(2n - \frac{1}{2}\right)\pi} + \mathcal{O}(n^{-1}). \tag{3.25}$$

Substituting this in (3.20), we obtain the asymptotic expression for x_n:

$$\begin{aligned} x_n &= \ln\sqrt{b} + \ln(-\sin y_n) - \ln y_n \\ &= \ln\sqrt{b} + \ln\left[-\sin\left[\left(2n - \frac{1}{2}\right)\pi - \frac{\ln\left(2n - \frac{1}{2}\right)\pi}{\left(2n - \frac{1}{2}\right)\pi} + \mathcal{O}(n^{-1})\right]\right] \end{aligned}$$

$$- \ln\left[\left(2n-\frac{1}{2}\right)\pi - \frac{\ln\left(2n-\frac{1}{2}\right)\pi}{\left(2n-\frac{1}{2}\right)\pi} + \mathcal{O}(n^{-1})\right]$$

$$= \ln\sqrt{b} - \ln\left(2n-\frac{1}{2}\right)\pi + \mathcal{O}(n^{-1}).$$

Finally, we obtain the asymptotic expression $\xi_n = x_n + iy_n$ given by (3.13). The proof is complete. □

The same arguments allows one to deduce the asymptotic roots of $f_2(\lambda)$ and we just characterize these properties as the following proposition without supplying the proof.

Proposition 3.2. *Let $b > 0$ and let $f_2(\lambda)$ be given by* (3.11). *Then*

$$f_2(\lambda) = \lambda e^{\lambda} + \sqrt{b}$$

has the roots

$$\sigma(f_2(\lambda)) = \{\eta_n,\ \overline{\eta_n}\}_{n\in\mathbb{N}} \cup \{\nu_{2j}\}, \quad j \in \mathcal{I}_1 \subseteq \{1,2\}, \tag{3.26}$$

where ν_{2j} are possible real roots of $f_2(\lambda)$ and $\mathcal{I}_1$ is an empty subset or a subset with at most finitely two numbers. More precisely,

(i) *if $b = e^{-2}$, then $f_2(\lambda)$ has only one real root $\nu = -1$;*
(ii) *if $b > e^{-2}$, then $f_2(\lambda)$ has no real roots;*
(iii) *if $0 < b < e^{-2}$, then $f_2(\lambda)$ have two negative real roots.*

Moreover, η_n has the following asymptotic expression:

$$\eta_n = \left[\ln\sqrt{b} - \ln\left(2n-\frac{3}{2}\right)\pi\right] + i\left[\left(2n-\frac{3}{2}\right)\pi - \frac{\ln\left(2n-\frac{3}{2}\right)\pi}{\left(2n-\frac{3}{2}\right)\pi}\right] + \mathcal{O}(n^{-1}). \tag{3.27}$$

Remark 3.1. *From* (3.13) *and* (3.27), *we obtain the asymptotic expressions of the eigenvalues $\{\xi_n, \overline{\xi_n}, \eta_n, \overline{\eta_n}\}$. It is found that the real part of high frequencies approaches $-\infty$ as n tends to ∞. Moreover, high frequencies are less and less undamped with the feedback b.*

In summary, collecting Lemmas 3.1–3.4 and Propositions 3.1–3.2, we obtain the following spectrum distribution of $\mathcal{A}$.

Theorem 3.1. *Let $\mathcal{A}$ be given by* (2.5) *and* (2.6) *and let the condition* (3.1) *hold. Then we have the following conclusions for the spectrum of $\mathcal{A}$:*

(i) *for each $\lambda \in \sigma(\mathcal{A})$, we have* $\mathrm{Re}(\lambda) < 0$;

(ii) *$\mathcal{A}$ has infinitely many eigenvalues λ_n, $n \in \mathbb{N}$, in $\mathbb{C}^-$, and* $\operatorname{Re} \lambda_n \to -\infty$ *as $n \to \infty$*;
(iii) *$\mathcal{A}$ has only at most four real eigenvalues*;
(iv) *the spectrum $\sigma(\mathcal{A})$ has the following form*:

$$\sigma(\mathcal{A}) = \{\mu_i,\ i \in \mathcal{I}_2\} \cup \{\xi_n,\ \overline{\xi_n}\}_{n\in\mathbb{N}} \cup \{\eta_n,\ \overline{\eta_n}\}_{n\in\mathbb{N}}, \tag{3.28}$$

where μ_i denotes the real eigenvalue of $\mathcal{A}$, $\mathcal{I}_2 \subset \{1,2,3,4\}$ denotes that there are at most finitely four numbers, and ξ_n and η_n are complex eigenvalues which have the asymptotic expressions by (3.13) *and* (3.27), *respectively*;
(v) *each eigenvalue of $\mathcal{A}$ is simple except at most four eigenvalues.*

Proof. Since $\lambda \in \sigma(\mathcal{A})$ if and only if λ is a root of $h(\lambda)$ given by (3.2), assertions (i)–(iii) can be obtained directly from Lemmas 3.1–3.4. Moreover, by the Rouché theorem, $f(\lambda)$ given by (3.7) and $\tilde{f}(\lambda)$ given by (3.9) have the same asymptotic root expressions. Since $h(\lambda)$ and $f(\lambda)$ have the same roots, by Propositions 3.1–3.2, assertion (iv) follows.

Recall (see Lemma 2.2) that each eigenvalue λ of $\mathcal{A}$ is geometrically simple. Furthermore, it follows from a general formula of [11, p. 148] that

$$m_{(a)}(\lambda) \le p_\lambda \cdot m_{(g)}(\lambda) = p_\lambda,$$

where p_λ denotes the order of the pole of $R(\lambda, \mathcal{A})$ at λ, $m_{(a)}(\lambda)$ denotes the algebraic multiplicity of λ, and $m_{(g)}(\lambda)$ denotes the geometric multiplicity of λ. Expression (2.12) for $R(\lambda, \mathcal{A})$ implies that p_λ does not exceed the multiplicity of $\det \Delta(\lambda)$ at λ. Since we have proved (see Lemma 3.2) that each root of $\det \Delta(\lambda)$ is simple except for at most four values, we conclude assertion (v). The proof is complete. □

4. Non-basis property of eigenfunctions

In this section, we show that generalized eigenfunctions of $\mathcal{A}$ cannot form a Riesz basis in the Hilbert state space $\mathcal{H}$ via estimating the norm of the Riesz spectrum projection. First, we obtain the adjoint operator $\mathcal{A}^*$ of $\mathcal{A}$, in the following lemma.

Lemma 4.1. *Let $\mathcal{A}$ be defined by* (2.5) *and* (2.6). *Then its adjoint operator $\mathcal{A}^*$ has the following form*:

$$\mathcal{A}^* \begin{pmatrix} y \\ g \end{pmatrix} = \begin{pmatrix} A_0^T & \delta_0 \\ 0 & -\dfrac{d}{ds} \end{pmatrix} \begin{pmatrix} y \\ g \end{pmatrix} \quad \forall \begin{pmatrix} y \\ g \end{pmatrix} \in D(\mathcal{A}^*), \tag{4.1}$$

where

$$D(\mathcal{A}^*) = \Big\{ (y,g)^T \in \mathcal{H} \ \Big| \ g \in H^1\left([-2,-1) \cup (-1,0],\ \mathbb{C}^2\right), \quad g(-2) = A_2^T y,\ g(-1^+) - g(-1^-) = A_1^T y \Big\},$$

$\delta_0 g = g(0)$ for all $g \in C[-2,0]$, and A_i^T, $i = 0,1,2$, are the transposes of A_i.

Proof. For each $X = (x,f)^T \in D(\mathcal{A})$ and $Y = (y,g)^T \in D(\mathcal{A}^*)$, a direct computation yields

$$\begin{aligned}
&\langle \mathcal{A}X, Y\rangle_{\mathcal{H}} \\
&= \langle A_0x + A_1f(-1) + A_2f(-2), y\rangle_{\mathbb{C}^2} + \int_{-2}^{0} \left\langle \frac{d}{ds}f(s), g(s)\right\rangle_{\mathbb{C}^2} ds \\
&= \langle A_0x, y\rangle_{\mathbb{C}^2} + \langle A_1f(-1), y\rangle_{\mathbb{C}^2} + \langle A_2f(-2), y\rangle_{\mathbb{C}^2} \\
&+ \langle f(s), g(s)\rangle_{\mathbb{C}^2}\Big|_{-1}^{0} + \langle f(s), g(s)\rangle_{\mathbb{C}^2}\Big|_{-2}^{-1} - \int_{-2}^{0} \left\langle f(s), \frac{d}{ds}g(s)\right\rangle_{\mathbb{C}^2} ds \\
&= \left\langle x, A_0^T y\right\rangle_{\mathbb{C}^2} + \left\langle f(-1), A_1^T y\right\rangle_{\mathbb{C}^2} + \left\langle f(-2), A_2^T y\right\rangle_{\mathbb{C}^2} \\
&+ \langle f(0), g(0)\rangle_{\mathbb{C}^2} - \left\langle f(-1), g(-1^+)\right\rangle_{\mathbb{C}^2} + \left\langle f(-1), g(-1^-)\right\rangle_{\mathbb{C}^2} \\
&- \langle f(-2), g(-2)\rangle_{\mathbb{C}^2} + \int_{-2}^{0} \left\langle f(s), -\frac{d}{ds}g(s)\right\rangle_{\mathbb{C}^2} ds \\
&= \left\langle x, A_0^T y + g(0)\right\rangle_{\mathbb{C}^2} + \left\langle f(-1), A_1^T y + g(-1^-) - g(-1^+)\right\rangle_{\mathbb{C}^2} \\
&+ \left\langle f(-2), A_2^T y - g(-2)\right\rangle_{\mathbb{C}^2} + \int_{-2}^{0} \left\langle f(s), -\frac{d}{ds}g(s)\right\rangle_{\mathbb{C}^2} ds \\
&= \langle X, \mathcal{A}^*Y\rangle_{\mathcal{H}},
\end{aligned}$$

and hence we obtain $\mathcal{A}^*$ that is given by (4.1). The proof is complete. □

Lemma 4.2. *Let $\mathcal{A}^*$ be given by (4.1). Then the spectrum of $\mathcal{A}^*$ is*

$$\sigma(\mathcal{A}^*) = \sigma_p(\mathcal{A}^*) = \overline{\sigma(\mathcal{A})} = \{\overline{\lambda} \in \mathbb{C} | \det \Delta(\lambda)^* = 0\},$$

where

$$\Delta(\lambda)^* = \overline{\lambda} - A_0^T - A_1^T e^{-\overline{\lambda}} - A_2^T e^{-2\overline{\lambda}}.$$

Moreover, each $\overline{\lambda} \in \sigma(\mathcal{A}^*)$ is geometrically simple with the corresponding eigenfunction

$$\psi_{\overline{\lambda}} = (y, g(s)y)^T, \tag{4.2}$$

where $y = \left(\overline{\lambda}, 1\right)^T$ and

$$g(s) = \begin{cases} (A_1^T + A_2^T e^{-\overline{\lambda}})e^{-\overline{\lambda}(1+s)}, & s \in (-1,0], \\ A_2^T e^{-\overline{\lambda}(2+s)}, & s \in [-2,-1). \end{cases}$$

Proof. The first assertion is obvious and we only need to find an eigenfunction of $\mathcal{A}^*$. Let $\overline{\lambda} \in \sigma(\mathcal{A}^*)$ and let $\psi = (y, g)^T \in D(\mathcal{A}^*)$ be an eigenfunction of $\mathcal{A}^*$ corresponding to $\overline{\lambda}$. Then we have $\mathcal{A}^*\psi = \overline{\lambda}\psi$, which yields

$$\begin{cases} A_0^T y + g(0) = \overline{\lambda} y, \\ -g'(s) = \overline{\lambda} g(s), \quad s \in (-1, 0], \\ -g'(s) = \overline{\lambda} g(s), \quad s \in [-2, -1), \\ g(-1^+) - g(-1^-) = A_1^T y, \\ g(-2) = A_2^T y. \end{cases} \tag{4.3}$$

When $s \in [-2, -1)$, by the third and fifth equations of (4.3), we have

$$g(s) = A_2^T y e^{-\overline{\lambda}(s+2)}, \quad s \in [-2, -1). \tag{4.4}$$

Thus, by the fourth equation of (4.3), we have

$$g(-1^+) = A_1^T y + A_2^T y e^{-\overline{\lambda}}.$$

This and the second equation of (4.3) yield

$$g(s) = \left[A_1^T y + A_2^T y e^{-\overline{\lambda}}\right] e^{-\overline{\lambda}(s+1)}, \quad s \in (-1, 0]. \tag{4.5}$$

Substituting this into the first equation of (4.3), we conclude

$$A_0^T y + \left[A_1^T y + A_2^T y e^{-\overline{\lambda}}\right] e^{-\overline{\lambda}} = \overline{\lambda} y$$

and

$$\Delta(\lambda)^* y \doteq (\overline{\lambda} - A_0^T - A_1^T e^{-\overline{\lambda}} - A_2^T e^{-2\overline{\lambda}}) y$$
$$= \begin{pmatrix} \overline{\lambda} & k - ae^{-\overline{\lambda}} - be^{-2\overline{\lambda}} \\ -1 & \overline{\lambda} \end{pmatrix} y = 0.$$

Therefore, $y = (\bar{\lambda}, 1)^T$ is a nontrivial solution of the above equation. The proof is complete. □

Now we are in a position to show that the generalized eigenfunctions of $\mathcal{A}$ does not form a Riesz basis in $\mathcal{H}$.

Theorem 4.1. *Let $\mathcal{A}$ be defined by* (2.5) *and* (2.6)*, let $\lambda \in \sigma(\mathcal{A})$ be a simple eigenvalue of $\mathcal{A}$, and let $E(\lambda; \mathcal{A})$ be its Riesz spectrum projection. Then ϕ_λ and $\psi_{\overline{\lambda}}$ given by* (2.13) *and* (4.2)*, respectively, are two eigenfunctions of $\mathcal{A}$ and $\mathcal{A}^*$ with respect to λ, and when $\langle \phi_\lambda, \psi_{\overline{\lambda}} \rangle_{\mathcal{H}} = 1$, for each $X \in \mathcal{H}$, we have*

$$E(\lambda; \mathcal{A})X = \langle X, \psi_{\overline{\lambda}} \rangle_{\mathcal{H}} \phi_\lambda.$$

Furthermore, when $\operatorname{Re}\lambda \to -\infty$*, the Riesz spectrum projection $E(\lambda; \mathcal{A})$ has the following approximate estimation*:

$$\|E(\lambda; \mathcal{A})\| \approx \frac{|\lambda| e^{-\mu}}{2|\mu|} \to +\infty,$$

where

$$\mu = \lambda + \overline{\lambda} = 2\,\mathrm{Re}\,\lambda.$$

Therefore, it is impossible to obtain a uniform upper bound for the norms of the Riesz spectrum projection of $\mathcal{A}$, and hence the generalized eigenfunctions of $\mathcal{A}$ do not form a Riesz basis in the Hilbert space $\mathcal{H}$.

Proof. The first two assertions are obvious. Now we estimate the norm of $\|E(\lambda;\mathcal{A})\|$ as $\mathrm{Re}\,\lambda \to -\infty$. By Lemmas 2.2 and 4.1, we assume that

$$x_\lambda = k_1(\lambda)\,(1,\lambda)^T, \quad y_{\overline{\lambda}} = k_2(\lambda)\left(\overline{\lambda},1\right)^T,$$

where $k_1(\lambda), k_2(\lambda) \in \mathbb{C}$ are two coefficients to be determined so that

$$\langle \phi_\lambda, \psi_{\overline{\lambda}} \rangle_{\mathcal{H}} = 1.$$

So, a direct calculation yields

$$\begin{aligned}
1 = \langle \phi_\lambda, \psi_{\overline{\lambda}} \rangle_{\mathcal{H}} &= \langle x_\lambda, y_{\overline{\lambda}} \rangle_{\mathbb{C}^2} + \int_{-2}^{0} \left\langle e^{\lambda s} x_\lambda, g(s) \right\rangle_{\mathbb{C}^2} ds \\
&= \langle x_\lambda, y_{\overline{\lambda}} \rangle_{\mathbb{C}^2} + \int_{-1}^{0} \left\langle e^{\lambda s} x_\lambda, \left[A_1^T y_{\overline{\lambda}} + A_2^T e^{-\overline{\lambda}} y_{\overline{\lambda}}\right] e^{-\overline{\lambda}(s+1)} \right\rangle_{\mathbb{C}^2} ds \\
&\quad + \int_{-2}^{-1} \left\langle e^{\lambda s} x_\lambda, A_2^T y_{\overline{\lambda}} e^{-\overline{\lambda}(s+2)} \right\rangle_{\mathbb{C}^2} ds \\
&= \langle x_\lambda, y_{\overline{\lambda}} \rangle_{\mathbb{C}^2} + \int_{-1}^{0} e^{-\lambda} \left\langle x_\lambda, \left(A_1^T + A_2^T e^{-\overline{\lambda}}\right) y_{\overline{\lambda}} \right\rangle_{\mathbb{C}^2} ds \\
&\quad + \int_{-2}^{-1} e^{-2\lambda} \left\langle x_\lambda, A_2^T y_{\overline{\lambda}} \right\rangle_{\mathbb{C}^2} ds \\
&= k_1(\lambda)\overline{k_2(\lambda)}\left(2\lambda + a e^{-\lambda} + 2b e^{-2\lambda}\right).
\end{aligned}$$

Introduce the notation

$$\eta(\lambda) = 2\lambda + a e^{-\lambda} + 2b e^{-2\lambda}, \quad \mu = \lambda + \overline{\lambda} = 2\,\mathrm{Re}\,\lambda,$$
$$k_1(\lambda) = \sqrt{|\mathrm{Re}\,\lambda|}\, e^{\lambda}, \quad k_2(\lambda) = \frac{1}{\sqrt{|\mathrm{Re}\,\lambda|}\, e^{\overline{\lambda}} \cdot \overline{\eta(\lambda)}}.$$

Then we have

$$\langle \phi_\lambda, \psi_{\overline{\lambda}} \rangle_{\mathcal{H}} = 1.$$

Furthermore, if $\mathrm{Re}\,\lambda \to -\infty$, we have

$$\|\phi_\lambda\|^2_{\mathcal{H}} = \langle x_\lambda, x_\lambda\rangle_{\mathbb{C}^2} + \int\limits_{-2}^{0} \left\langle e^{\lambda s} x_\lambda, e^{\lambda s} x_\lambda \right\rangle_{\mathbb{C}^2} ds$$

$$= \langle x_\lambda, x_\lambda\rangle_{\mathbb{C}^2} + \int\limits_{-2}^{0} e^{\lambda s} \cdot e^{\overline{\lambda} s} \langle x_\lambda, x_\lambda\rangle_{\mathbb{C}^2} ds$$

$$= |k_1(\lambda)|^2 \left(1 + |\lambda|^2\right) \left(1 + \int\limits_{-2}^{0} e^{\mu s} ds\right)$$

$$= \frac{|\mu|}{2} e^{\mu} \left(1 + |\lambda|^2\right) \left(1 + \frac{1}{\mu} + \frac{e^{-2\mu}}{-\mu}\right) \approx \frac{|\lambda|^2 e^{-\mu}}{2} \to +\infty$$

and

$$\|\psi_{\overline{\lambda}}\|^2_{\mathcal{H}} = \|y_{\overline{\lambda}}\|^2_{\mathbb{C}^2} + \int\limits_{-2}^{0} \left\langle g(s) y_{\overline{\lambda}}, g(s) y_{\overline{\lambda}} \right\rangle_{\mathbb{C}^2} ds$$

$$= |k_2(\lambda)|^2 \left(1 + |\lambda|^2\right) + \int\limits_{-2}^{-1} \left\langle A_2^T e^{-\overline{\lambda}(s+2)} y_{\overline{\lambda}}, A_2^T e^{-\overline{\lambda}(s+2)} y_{\overline{\lambda}} \right\rangle_{\mathbb{C}^2} ds$$

$$+ \int\limits_{-1}^{0} \left\langle \left(A_1^T + A_2^T e^{-\overline{\lambda}}\right) e^{-\overline{\lambda}(s+1)} y_{\overline{\lambda}}, \left(A_1^T + A_2^T e^{-\overline{\lambda}}\right) e^{-\overline{\lambda}(s+1)} y_{\overline{\lambda}} \right\rangle_{\mathbb{C}^2} ds$$

$$= |k_2(\lambda)|^2 \left(1 + |\lambda|^2\right)$$

$$+ \int\limits_{-1}^{0} e^{-\mu(s+1)} |k_2(\lambda)|^2 \left[a^2 + b^2 e^{-\mu} + ab\left(e^{-\lambda} + e^{-\overline{\lambda}}\right)\right] ds$$

$$+ \int\limits_{-2}^{-1} e^{-\mu(s+2)} |k_2(\lambda)|^2 b^2 ds = |k_2(\lambda)|^2$$

$$\times \left[1 + |\lambda|^2 + \left(a^2 + b^2 + b^2 e^{-\mu} + ab\left(e^{-\lambda} + e^{-\overline{\lambda}}\right)\right) \cdot \frac{1}{-\mu}\left(e^{-\mu} - 1\right)\right]$$

$$\approx |k_2(\lambda)|^2 \left[|\lambda|^2 + \left(a^2 + b^2 + b^2 e^{-\mu} + ab\left(e^{-\lambda} + e^{-\overline{\lambda}}\right)\right) \cdot \frac{e^{-\mu}}{-\mu}\right]$$

$$= \frac{|\lambda|^2 + \left(a^2 + b^2 + b^2 e^{-\mu} + ab\left(e^{-\lambda} + e^{-\overline{\lambda}}\right)\right) \cdot \frac{e^{-\mu}}{-\mu}}{\left|\frac{\mu}{2}\right| e^{\mu} \cdot \left|2\lambda + a e^{-\lambda} + 2b e^{-2\lambda}\right|^2}$$

$$= \frac{2e^{-\mu}\left[|\lambda|^2|\mu| + \left(a^2+b^2+b^2e^{-\mu}+ab\left(e^{-\lambda}+e^{-\overline{\lambda}}\right)\right)\cdot e^{-\mu}\right]}{|\mu|^2\cdot|2\lambda+ae^{-\lambda}+2be^{-2\lambda}|^2}$$
$$\approx \frac{2e^{-\mu}\left[|\lambda|^2|\mu| + \left(a^2+b^2+b^2e^{-\mu}+ab\left(e^{-\lambda}+e^{-\overline{\lambda}}\right)\right)\cdot e^{-\mu}\right]}{|\mu|^2\cdot\left[2|\lambda|+ae^{-\frac{\mu}{2}}+2be^{-\mu}\right]^2}$$
$$\approx \frac{2b^2e^{-3\mu}}{4b^2e^{-2\mu}|\mu|^2} = \frac{e^{-\mu}}{2|\mu|^2} \to +\infty.$$

Finally, we eventually obtain the approximate estimate of $\|E(\lambda;\mathcal{A})\|$:

$$\|E(\lambda;\mathcal{A})\| = \|\phi_\lambda\|_{\mathcal{H}}\|\psi_{\overline{\lambda}}\|_{\mathcal{H}} \approx \frac{|\lambda|e^{-\mu}}{2|\mu|} \to +\infty \quad \text{as } \operatorname{Re}\lambda \to -\infty.$$

Hence, there is no uniformly upper bound for $\|E(\lambda;\mathcal{A})\|$ as $\operatorname{Re}\lambda \to -\infty$. Therefore, the generalized eigenfunctions of $\mathcal{A}$ do not form a Riesz basis in Hilbert space $\mathcal{H}$. The proof is complete. □

5. Spectrum-determined growth condition and exponential stability

Now we are in a position to consider the spectrum-determined growth condition for system (2.8), which is one of the most difficult problems for infinite-dimensional systems. Our proof is based on the following characterization condition (see [11, Corollary 3.40]) and this method was used by the authors to treat the heat system with memory (see [19]).

Lemma 5.1. *Let $T(t)$ be a C_0-semigroup on a Hilbert space $\mathbf{H}$ with its generator $\mathbf{A}$. Let $\omega(\mathbf{A})$ be the growth bound of $T(t)$ and*

$$s(\mathbf{A}) := \sup\left\{\operatorname{Re}\lambda \;\middle|\; \lambda\in\sigma(\mathbf{A})\right\}$$

be the spectral bound of $\mathbf{A}$. Then

$$\omega(\mathbf{A}) = \inf\left\{\omega > s(\mathbf{A}) \;\middle|\; \sup_{\tau\in\mathbb{R}}\|R(\sigma+i\tau,\mathbf{A})\| < M_\sigma < \infty\ \forall\sigma\geq\omega\right\}.$$

We also need [14, Lemma 1.2] (see also [9]).

Lemma 5.2. *Let*

$$D(\lambda) = 1 + \sum_{i=1}^{n} Q_i(\lambda)e^{\alpha_i\lambda},$$

where Q_i are polynomials of λ, α_i are complex numbers, and n is a positive integer. Then for all λ lying outside circles of radii $\varepsilon > 0$ centered at the roots of $D(\cdot)$, we have

$$|D(\lambda)| \geq C(\varepsilon) > 0$$

for some constant $C(\varepsilon)$ depending only on ε.

Theorem 5.1. *Let $\mathcal{A}$ be given by* (2.5) *and* (2.6)*. Then the spectrum-determined growth condition holds for $e^{\mathcal{A}t}$, i.e., $s(\mathcal{A}) = \omega(\mathcal{A})$.*

Proof. By Lemma 5.1, the proof will be accomplished if we can show that for any $\lambda \neq 0$ and $\lambda = \alpha + i\beta$ with $\alpha \geq \omega > s(\mathcal{A})$ and $\beta \in \mathbb{R}$, there is a constant M_α such that

$$\sup_{\beta \in \mathbb{R}} \|R(\alpha + i\beta, \mathcal{A})\| \leq M_\alpha < \infty. \tag{5.1}$$

Let $\lambda = \alpha + i\beta \in \mathbb{C}$, where $\alpha \geq \omega > s(\mathcal{A})$ and $\beta \in \mathbb{R}$. Then $\lambda \in \rho(\mathcal{A})$. By Lemma 2.2, for all $Y = (y, g)^T \in \mathcal{H}$, there exists $X = R(\lambda, \mathcal{A})Y = (x, f)^T \in D(\mathcal{A})$ given by (2.12). For convenience, we rewrite it here:

$$x = \Delta(\lambda)^{-1} \left[y + A_1 \int_{-1}^{0} e^{-\lambda(1+s)} g(s)ds + A_2 \int_{-2}^{0} e^{-\lambda(2+s)} g(s)ds \right],$$

$$f(s) = e^{\lambda s} x + \int_{s}^{0} e^{\lambda(s-r)} g(r)dr.$$

Since $\lambda \in \rho(\mathcal{A})$ and $\det \Delta(\lambda) \neq 0$, we have

$$\Delta(\lambda)^{-1} = \begin{pmatrix} \dfrac{\lambda}{\lambda^2 + k - ae^{-\lambda} - be^{-2\lambda}} & \dfrac{1}{\lambda^2 + k - ae^{-\lambda} - be^{-2\lambda}} \\ \dfrac{-k + ae^{-\lambda} + be^{-2\lambda}}{\lambda^2 + k - ae^{-\lambda} - be^{-2\lambda}} & \dfrac{\lambda}{\lambda^2 + k - ae^{-\lambda} - be^{-2\lambda}} \end{pmatrix}. \tag{5.2}$$

Since $\|A_1\| = |a|$, $\|A_2\| = |b|$, we have

$$\begin{aligned} \|\Delta(\lambda)^{-1}\| &= \frac{2|\lambda| + 1 + |k - ae^{-\lambda} - be^{-2\lambda}|}{|\lambda^2 + k - ae^{-\lambda} - be^{-2\lambda}|} \\ &\leq \frac{2|\lambda| + 1 + |k| + |ae^{-\lambda}| + |b|e^{-2\lambda}}{|\lambda^2 + k - ae^{-\lambda} - be^{-2\lambda}|} \\ &= \frac{2 + \dfrac{1 + |k|}{\sqrt{\alpha^2 + \beta^2}} + \dfrac{|a|e^{-\alpha}}{\sqrt{\alpha^2 + \beta^2}} + \dfrac{be^{-2\alpha}}{\sqrt{\alpha^2 + \beta^2}}}{\left| \lambda + \dfrac{k}{\lambda} - \dfrac{ae^{-\lambda} + be^{-2\lambda}}{\lambda} \right|}. \end{aligned}$$

Also by Lemma 2.2, we have

$$s(A) = \sup\{\operatorname{Re} \lambda \mid \lambda \in \sigma(\mathcal{A})\} = \sup\{\operatorname{Re} \lambda \mid \lambda \in \sigma_p(\mathcal{A})\} = \sup\{\operatorname{Re} \lambda \mid \det \Delta(\lambda) = 0\}.$$

Introduce the notation

$$\varepsilon_\alpha = \inf_{\substack{\lambda_n \in \sigma_p(\mathcal{A}), \\ \beta \in \mathbb{R}}} |\lambda_n - \alpha - i\beta|.$$

By Lemma 5.2, there is a positive constant $C(\varepsilon_\alpha)$ depending on α such that

$$\left|\lambda + \frac{k}{\lambda} - \frac{ae^{-\lambda} + be^{-2\lambda}}{\lambda}\right| \geq C(\varepsilon_\alpha) > 0.$$

Therefore, there exists a positive constant $M_{1\alpha} > 0$ depending on α such that

$$\sup_{\beta\in\mathbb{R}} \|\Delta(\lambda)^{-1}\| \leq M_{1\alpha} < \infty.$$

Due to the estimates

$$\int_{-1}^{0} e^{-\lambda(1+s)} e^{-\overline{\lambda}(1+s)} ds = \int_{-1}^{0} e^{-2\alpha(1+s)} ds = \frac{1-e^{-2\alpha}}{2\alpha},$$

$$\int_{-2}^{0} e^{-\lambda(2+s)} e^{-\overline{\lambda}(2+s)} ds = \int_{-2}^{0} e^{-2\alpha(2+s)} ds = \frac{1-e^{-4\alpha}}{2\alpha},$$

$$\int_{-2}^{0} e^{\lambda s} e^{\overline{\lambda} s} ds = \int_{-2}^{0} e^{2\alpha s} ds = \frac{1-e^{-4\alpha}}{2\alpha},$$

$$\int_{s}^{0} e^{\lambda(s-r)} e^{\overline{\lambda}(s-r)} dr = \int_{s}^{0} e^{2\alpha(s-r)} dr = \frac{1-e^{2\alpha s}}{2\alpha},$$

$$\int_{-2}^{0} \left(\frac{1-e^{2\alpha s}}{2\alpha}\right) ds = \frac{1}{\alpha} - \frac{1-e^{-4\alpha}}{4\alpha^2},$$

there exist two positive constant numbers $M_{2\alpha}$ and $M_{3\alpha}$ depending on α such that

$$\sup_{\beta\in\mathbb{R}} \int_{-1}^{0} |e^{-\lambda(1+s)}|^2 ds \leq M_{2\alpha} < \infty,$$

$$\sup_{\beta\in\mathbb{R}} \int_{-2}^{0} |e^{-\lambda(2+s)}|^2 ds \leq M_{2\alpha} < \infty,$$

$$\sup_{\beta\in\mathbb{R}} \int_{-2}^{0} |e^{\lambda s}|^2 ds \leq M_{2\alpha} < \infty,$$

$$\sup_{\beta\in\mathbb{R}} \left|\int_{-2}^{0} \left(\frac{1-e^{2\alpha s}}{2\alpha}\right) ds\right| \leq M_{3\alpha} < \infty.$$

Hence, we have

$$\sup_{\beta\in\mathbb{R}}\|x\|^2_{\mathbb{C}^2}$$

$$= \sup_{\beta\in\mathbb{R}}\left\|\Delta(\lambda)^{-1}\left[y + A_1\int_{-1}^{0} e^{-\lambda(1+s)}g(s)ds + A_2\int_{-2}^{0} e^{-\lambda(2+s)}g(s)ds\right]\right\|^2_{\mathbb{C}^2}$$

$$\leq 3\left(\sup_{\beta\in\mathbb{R}}\|\Delta(\lambda)^{-1}\|\right)^2\left(\|y\|^2_{\mathbb{C}^2} + a^2\sup_{\beta\in\mathbb{R}}\left\|\int_{-1}^{0} e^{-\lambda(1+s)}g(s)ds\right\|^2_{\mathbb{C}^2}\right.$$

$$\left.+b^2\sup_{\beta\in\mathbb{R}}\left\|\int_{-2}^{0} e^{-\lambda(2+s)}g(s)ds\right\|^2_{\mathbb{C}^2}\right) \leq 3\left(\sup_{\beta\in\mathbb{R}}\|\Delta(\lambda)^{-1}\|\right)^2$$

$$\times\left[\|y\|^2_{\mathbb{C}^2} + a^2\left(\sup_{\beta\in\mathbb{R}}\int_{-1}^{0}|e^{-\lambda(1+s)}|^2ds\right)\left(\int_{-1}^{0}\|g(s)\|^2_{\mathbb{C}^2}ds\right)\right.$$

$$\left.+b^2\left(\sup_{\beta\in\mathbb{R}}\int_{-2}^{0}|e^{-\lambda(2+s)}|^2ds\right)\left(\int_{-2}^{0}\|g(s)\|^2_{\mathbb{C}^2}ds\right)\right]$$

$$\leq 3M^2_{1\alpha}\|y\|^2_{\mathbb{C}^2} + 3M^2_{1\alpha}M_{2\alpha}\left(a^2+b^2\right)\int_{-2}^{0}\|g(s)\|^2_{\mathbb{C}^2}ds$$

and

$$\sup_{\beta\in\mathbb{R}}\int_{-2}^{0}\|f(s)\|^2_{\mathbb{C}^2}ds = \sup_{\beta\in\mathbb{R}}\int_{-2}^{0}\left\|e^{\lambda s}x + \int_{s}^{0} e^{\lambda(s-r)}g(r)dr\right\|^2_{\mathbb{C}^2}ds$$

$$\leq 2\sup_{\beta\in\mathbb{R}}\int_{-2}^{0}\|e^{\lambda s}x\|^2_{\mathbb{C}^2}ds + 2\sup_{\beta\in\mathbb{R}}\int_{-2}^{0}\left\|\int_{s}^{0} e^{\lambda(s-r)}g(r)dr\right\|^2_{\mathbb{C}^2}ds$$

$$\leq 2\|x\|^2_{\mathbb{C}^2}\sup_{\beta\in\mathbb{R}}\int_{-2}^{0}|e^{\lambda s}|^2ds + 2\sup_{\beta\in\mathbb{R}}\int_{-2}^{0}\left(\int_{s}^{0}|e^{\lambda(s-r)}|^2dr\right)\left(\int_{s}^{0}\|g(r)\|^2_{\mathbb{C}^2}dr\right)ds$$

$$\leq 2M_{2\alpha}\|x\|^2_{\mathbb{C}^2} + 2\int_{-2}^{0}\|g(s)\|^2_{\mathbb{C}^2}ds\sup_{\beta\in\mathbb{R}}\int_{-2}^{0}\left(\frac{1-e^{2\alpha s}}{2\alpha}\right)ds$$

$$\leq 2M_{2\alpha}\|x\|^2_{\mathbb{C}^2} + 2M_{3\alpha}\int_{-2}^{0}\|g(s)\|^2_{\mathbb{C}^2}ds$$

$$\leq 6M_{1\alpha}^2 M_{2\alpha}\|y\|_{\mathbb{C}^2}^2 + \left[6M_{1\alpha}^2 M_{2\alpha}^2(a^2+b^2) + 2M_{3\alpha}\right]\int_{-2}^{0}\|g(s)\|_{\mathbb{C}^2}^2 ds.$$

Therefore, there is a positive constant $M_\alpha > 0$ depending on α such that

$$\sup_{\beta\in\mathbb{R}}\|X\|_{\mathcal{H}}^2 = \sup_{\beta\in\mathbb{R}}\left\{\|x\|_{\mathbb{C}^2}^2 + \int_{-2}^{0}\|f(s)\|_{\mathbb{C}^2}^2 ds\right\}$$
$$\leq M_\alpha\left\{\|y\|_{\mathbb{C}^2}^2 + \int_{-2}^{0}\|g(s)\|_{\mathbb{C}^2}^2 ds\right\} = M_\alpha\|Y\|_{\mathcal{H}}^2 < \infty.$$

This yields

$$\sup_{\beta\in\mathbb{R}}\|X\|_{\mathcal{H}} \leq \sqrt{M_\alpha}\|Y\|_{\mathcal{H}} < \infty,$$

so (5.1) holds. The proof is complete. □

The following theorem gives a strongly exponential stability for system (2.8).

Theorem 5.2. *Let $k < 0$, let $\mathcal{A}$ be given by (2.5) and (2.6), and let condition (3.1) hold. Then $e^{\mathcal{A}t}$ generated by $\mathcal{A}$ is exponentially stable, i.e., there exist constants M and $\omega > 0$ such that*

$$\|e^{\mathcal{A}t}\| \leq Me^{-\omega t}.$$

Proof. By the spectrum-determined growth condition as established by Theorem 5.1, the verification of the stability for $e^{\mathcal{A}t}$ is determined by the spectral distribution of $\mathcal{A}$. From Theorem 3.1, for each $\lambda_n \in \sigma(\mathcal{A})$, we have $\operatorname{Re}\lambda_n \to -\infty$ as $n \to \infty$. Hence, $e^{\mathcal{A}t}$ is exponentially stable if and only if

$$\operatorname{Re}\lambda < 0 \quad \forall\lambda \in \sigma(\mathcal{A}).$$

This has been claimed from the first decision of Theorem 3.1. The proof is complete. □

6. Numerical applications

In this section, we give some numerical simulation results for an inverted pendulum with two delayed position feedback controls (see Fig. 2) that can be described by

$$\ddot{\theta}(t) - \frac{g}{l}\theta(t) = \hat{a}\theta(t-\tau) + \hat{b}\theta(t-2\tau), \tag{6.1}$$

where $\theta(t)$ is the angular displacement, l is the length of the pendulum, and g is the gravitational acceleration. Now we choose $g = 9.8\,\mathrm{m/s^2}$, $\tau = 0.143\,\mathrm{s}$, $l = 0.4\,\mathrm{m}$, $\hat{a} = -63.73$, $\hat{b} = 36.76$, and the initial conditions $\theta(0) = 1$ and $\theta'(0) = 0$. The simulation present the convergence of the state of system (6.1) by Fig. 3.

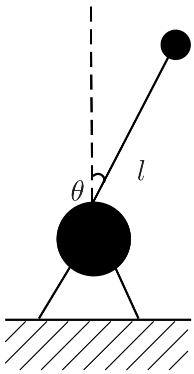

Fig. 2. Inverted pendulum

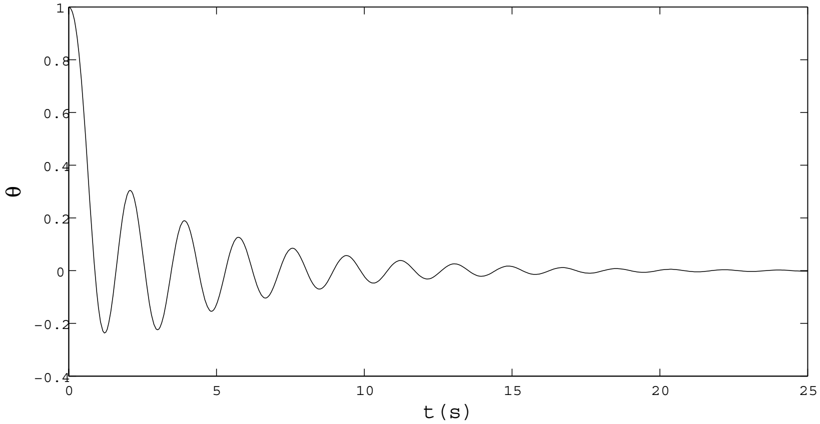

Fig. 3. Angular position of the inverted pendulum (6.1) under two delayed position feedback controls

References

1. F. M. Atay. Balancing the inverted pendulum using position feedback. *Appl. Math. Lett.* **12** (1999), 51–56.
2. R. Datko, J. Lagnese, and M. P. Polis. An example on the effect of time delays in boundary feedback stabilization of wave equations. *SIAM J. Control Optim.* **24** (1986), No. 1, 152–156.
3. B. Z. Guo, J. M. Wang, and K. Y. Yang. Dynamic stabilization of an Euler–Bernoulli beam under boundary control and non-collocated observation. *Systems Control Lett.* **57** (2008), No. 9, 740–749.
4. T. Hagen. Asymptotic solutions of characteristic equations. *Nonlinear Anal. Real World Appl.* **6** (2005), No. 3, 429–446.
5. Z. J. Han and G. Q. Xu. Dynamical behavior of a hybrid system of nonhomogeneous Timoshenko beam with partial non-collocated inputs. *J. Dynam. Control Systems* **17** (2011), No. 1, 77–121.
6. H. Y. Hu. Using delayed state feedback to stabilize periodic motions of an oscillator. *J. Sound Vibration* **275** (2004), Nos. 3–5, 1009–1025.
7. M. Krstic. Control of an unstable reaction-diffusion PDE with long input delay. *Systems Control Lett.* **58** (2009), Nos. 10–11, 773–782.
8. H. Kwakernaak and R. Sivan. Linear optimal control systems. *Wiley, New York* (1972).
9. R. E. Langer. On the zeros of exponential sum and integrals. *Bull. Am. Math. Soc.* **7** (1931), 213–239.
10. B. Liu and H. Hu. Stabilization of linear undamped systems via position and delayed position feedbacks. *J. Sound Vibration* **312** (2008), 509–528.
11. Z. H. Luo, B. Z. Guo, and O. Morgul. Stability and stabilization of infinite-dimensional systems with applications. *Springer-Verlag, London* (1999).
12. A. Pazy. Semigroup of linear operators and applications to partial differential equations. *Springer-Verlag, New York* (1983).
13. J. P. Richard. Time-delay systems: an overview of some recent advances and open problems. *Automat. J. IFAC* **39** (2003), No. 10, 1667–1694.
14. A. A. Shkalikov. Boundary-value problems for ordinary differential equations with a parameter in the boundary conditions. *J. Sov. Math.* **33** (1986), 1311–1342.
15. N. C. Singer and W. P. Seering. Preshaping command inputs to reduce system vibration. *ASME J. Dynam. Systems Measur. Control* **112** (1990), 76–82.
16. E. D. Sontag. Mathematical control theory. Deterministic finite-dimensional systems. *Springer-Verlag, New York* (1990).
17. I. H. Suh and Z. Bien. Proportional minus delay controller. *IEEE Trans. Automat. Control* **24** (1979), No. 2, 370–372.

18. ______, Use of time-delay actions in the controller design. *IEEE Trans. Automat. Control* **25** (1980), No. 3, 600–603.
19. J. M. Wang, B. Z. Guo, and M. Y. Fu. Dynamic behavior of a heat equation with memory. *Math. Methods Appl. Sci.* **32** (2009), No. 10, 1287–1310.
20. K. Youcef-Toumi and O. Ito. A time delay controller for systems with unknown dynamics. *ASME J. Dynam. Systems Measur. Control* **112** (1990), 133–142.

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