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Exponential stability and spectral analysis of a delayed ring neural network with a small-world connection

Dong-Xia Zhao · Jun-Min Wang

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Abstract This paper addresses the dynamical behavior of the linearized delayed ring neural network system with a small-world connection. The semigroup approach is adopted in investigation. The asymptotic eigenvalues of the system are presented. It shows that the spectrum of the system is located in the left half complex plane and its real part goes to $-\infty$ when the connection weights between neurons are well-defined. The spectrum determined growth condition is held true and the exponential stability of the system is then established. Moreover, we present the necessary conditions for the neuron and feedback gains, for which the closed-loop system is delay-independent exponentially stable, and we further provide the sufficient and necessary conditions when the concrete number of neurons and the location of small-world connection are given. Finally, numerical simulations are presented to illustrate the convergence of the state for the system and demonstrate the effect of the feedback gain on stability.

D.-X. Zhao (✉) · J.-M. Wang
Department of Mathematics,
Beijing Institute of Technology, Beijing 100081, PR China
e-mail: zhaodongxia6@sina.com

J.-M. Wang
e-mail: wangjc@graduate.hku.hk

D.-X. Zhao
Department of Mathematics, North University of China,
Taiyuan 030051, PR China

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1 Introduction

Since Hopfield in [3] introduced a continuous version of a circuit equation for a network of n neurons, a plenty of research in theory and application of neural networks are appeared (see [1, 11] and the references therein). With the pioneering work of Watts and Strogatz [17], small-world networks have caused great interest. Generally speaking, small-world network, like most of the networks in biology, technology, and social sciences, is intermediate between regular network and random network as well as a special type of complex network with a high degree of local clustering and a small average distance, which is obtained by randomly adding a small fraction of connection in an originally nearest-neighbor coupled network. Many common networks such as power grids, financial networks, internet servers, human communities, and disordered porous media, behave like small-world networks.

There are many research progress on neural networks. For examples, Li and Chen (see [8]) prove that Hopf bifurcation occurs in the small-world networks model with nonlinear interactions and time delays by choosing the nonlinear interaction strength as a bifurcation parameter; they also determine the stability of the bifurcating periodic solutions and the direction of the Hopf bifurcation. In [9], Li and Chen

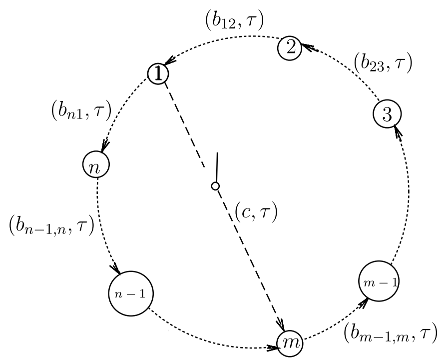

Fig. 1 Schematic of a ring network with a small-world connection

got a more exciting conclusion that neural networks with small-world connections are easier to be stabilized than the regular fully-connected counterparts. In addition, Hu, Xu, and Wang (see [4, 18, 20], and [19]) studied the dynamical behaviors of a neural network, such as asymptotic stability, global stability, Hopf bifurcation and chaos. Especially, in [19], Xu and Wang studied a delayed ring neural network with a small-world connection and found that the design of a small-world connection is a simple but efficient "switch" to control the dynamics of the system. However, these studies are mainly focused on the dynamics of neural network, and there are fewer concerning on the stability of neural networks with small-world connections (short-cuts). On the other hand, it will be exciting to investigate the stability of a neural network with small-world connections from another point of view, such as exponential stability, spectrum analysis, and stability interval of the small-world connection strength.

In this paper, we are interested in a ring neural network with a small world connection as shown in Fig. 1 (see [19]), which can be described by the so-called Hopfield first-order functional-differential system:

$$\dot{x_i}(t) = -kx_i(t) + \sum_{j=1}^{n} b_{ij} f\big(x_j(t-\tau)\big),\quad i = 1, 2, \dots, n, \tag{1.1}$$

where $x_i(t)$ is the neuron response, $k > 0$ is the neuron gains, $f(u) = \tanh(u)$ is the activation function of neurons, $\tau > 0$ is the time delay, and b_{ij} is the connection weight between neurons:

$$b_{ij} = \begin{cases} \neq 0, & i = 1, 2, \dots, n-1,\ j = i+1, \\ c, & i = m,\ j = 1, \\ \neq 0, & i = n,\ j = 1, \\ 0, & \text{otherwise.} \end{cases}$$

$b_{m1} = c \neq 0$ is the short-cut strength (for simplification, only one short-cut is included here), and define

$$B = \{b_{ij}\}_{i,j=1}^{n} = \begin{bmatrix} 0 & b_{12} & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & b_{23} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ c & 0 & 0 & \ddots & b_{m,m+1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & 0 & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \ddots & b_{(n-1)n} \\ b_{n1} & 0 & 0 & \cdots & 0 & \cdots & 0 \end{bmatrix}. \tag{1.2}$$

The aim of this paper is to give a more detailed spectral analysis of the system (1.1), and show that the spectrum determined growth condition is held. The exponential stability is then established. Finally, the delay-independent stability interval of c is also concluded. This is completely different from the existed study in the field of neural network.

The paper is organized as follows. In the next section, we formulate the system (1.1) into an abstract evolution equation and prove the well-posedness of the system by the semigroup approach. Section 3 is devoted to the detailed spectral analysis of the system. It is obtained that with some conditions required on the connection weights between neurons, all eigenvalues λ_n are located in the left half complex plane and their real parts $\mathrm{Re}\,\lambda_n$ go to $-\infty$ as $n \to \infty$. The asymptotic spectral expression is also presented. In Sect. 4, the spectrum determined growth condition is held true. In Sect. 5, the exponential stability of the system and the relation between stability and the value of c are established. Finally, some numerical simulations are presented in Sect. 6 to illustrate the eigenvalue distributions, the stability of the system, and the effect of c on stability.

2 Setup and well-posedness of the system

In this section, we shall convert the system (1.1) into an abstract evolution equation and then discuss the well-posedness of the system.

Since $f'(0)=1$, it results in the linearized equation of (1.1) at origin in the vectorizing form:

$$\dot{x}(t) = -kx(t) + Bx(t-\tau), \tag{2.1}$$

where $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$, and "$T$" denotes the transpose of a vector or a matrix. It is natural to set the initial date of (2.1) as the following:

$$\begin{cases} x(0) = x_0 = (x_{10}, x_{20}, \ldots, x_{n0}), \\ x(s) = \phi(s), \quad s \in [-\tau, 0], \end{cases} \tag{2.2}$$

where $x_0 \in \mathbb{C}^n$, $\phi \in L^2([-\tau,0],\mathbb{C}^n)$. We consider system (2.1) in the Hilbert state space

$$\mathcal{H} = \mathbb{C}^n \times L^2\big([-\tau,0],\mathbb{C}^n\big)$$

equipped with the usual inner product:

$$\langle X, Y\rangle_{\mathcal{H}} = \langle x, y\rangle_{\mathbb{C}^n} + \int_{-\tau}^{0} \big\langle f(s), g(s)\big\rangle_{\mathbb{C}^n} ds, \tag{2.3}$$

where $X=(x,f)^T \in \mathcal{H}$, $Y=(y,g)^T \in \mathcal{H}$. Define a linear operator $\mathcal{A}: \mathcal{H} \to \mathcal{H}$ by:

$$\mathcal{A}\begin{pmatrix} x \\ f \end{pmatrix} = \begin{pmatrix} -k & B\delta_1 \\ 0 & \frac{d}{ds} \end{pmatrix}\begin{pmatrix} x \\ f \end{pmatrix} \tag{2.4}$$

with

$$D(\mathcal{A}) = \big\{(x,f)^T \in \mathcal{H} \mid f \in H^1\big([-\tau,0],\mathbb{C}^n\big), \\ f(0)=x\big\}, \tag{2.5}$$

where $\delta_1 f = f(-\tau), \forall f \in C[-\tau,0]$ and B are given by (1.2).

Denote

$$\begin{cases} f(t,s) = x(t+s), \quad s \in [-\tau,0], \\ X(t) = (x(t), f(t,s))^T, \\ X(0) = X_0 := (x_0, \phi(s))^T, \quad s \in [-\tau,0], \end{cases} \tag{2.6}$$

then system (2.1) and (2.2) can be formulated into the following abstract evolution equation on $\mathcal{H}$:

$$\begin{cases} \frac{dX(t)}{dt} = \mathcal{A}X(t), \quad t>0, \\ X(0) = X_0. \end{cases} \tag{2.7}$$

Now we give the following two lemmas about the properties of $\mathcal{A}$.

Lemma 1 *Let $\mathcal{A}$ be given by (2.4) and (2.5), and let*

$$\langle X, Y\rangle_1 = \langle x, y\rangle_{\mathbb{C}^n} + \int_{-\tau}^{0} q(s)\big\langle f(s), g(s)\big\rangle_{\mathbb{C}^n} ds, \tag{2.8}$$

where $X=(x,f)^T \in \mathcal{H}$, $Y=(y,g)^T \in \mathcal{H}$, and

$$q(s) = \tau^{-2}\|B\|^2 s^2 + 1 > 0$$

is a bounded function in $s \in [-\tau,0]$. Then $\langle\cdot,\cdot\rangle_1$ is an inner product in $\mathcal{H}$ and it is equivalent to the general one given by (2.3). Moreover, there is a positive constant $M>0$ such that

$$\operatorname{Re}\langle \mathcal{A}X, X\rangle_1 \le M\langle X, X\rangle_1, \quad \forall X \in D(\mathcal{A}). \tag{2.9}$$

Hence, $\mathcal{A} - M$ is dissipative in $\mathcal{H}$.

Proof The first conclusion is obvious and we only need to show (2.9). For each $X=(x,f)^T \in D(\mathcal{A})$, it has

$$\langle \mathcal{A}X, X\rangle_1 = \big\langle -kx + Bf(-\tau), x\big\rangle_{\mathbb{C}^n} \\ + \int_{-\tau}^{0} q(s)\Big\langle \frac{d}{ds}f(s), f(s)\Big\rangle_{\mathbb{C}^n} ds.$$

A direct computation to yield

$$\begin{aligned} \operatorname{Re}\langle \mathcal{A}X, X\rangle_1 &\le -k\|x\|^2_{\mathbb{C}^n} + \|B\|\big\|f(-\tau)\big\|_{\mathbb{C}^n}\|x\|_{\mathbb{C}^n} \\ &\quad + \frac{1}{2}\int_{-\tau}^{0} q(s)\frac{d}{ds}\big\|f(s)\big\|^2_{\mathbb{C}^n} ds \\ &\le -k\|x\|^2_{\mathbb{C}^n} + \frac{1}{2}\big(\|B\|^2\big\|f(-\tau)\big\|^2_{\mathbb{C}^n} \\ &\quad + \|x\|^2_{\mathbb{C}^n}\big) \\ &\quad + \frac{1}{2}\bigg(q(s)\big\|f(s)\big\|^2_{\mathbb{C}^n}\Big|_{-\tau}^{0} \\ &\quad - \int_{-\tau}^{0} q'(s)\big\|f(s)\big\|^2_{\mathbb{C}^n} ds\bigg) \end{aligned}$$

$$= \left(-k+\frac{1}{2}\right)\|x\|^2_{\mathbb{C}^n} + \frac{1}{2}\|B\|^2\|f(-\tau)\|^2_{\mathbb{C}^n} + \frac{1}{2}q(0)\|f(0)\|^2_{\mathbb{C}^n} - \frac{1}{2}q(-\tau)\|f(-\tau)\|^2_{\mathbb{C}^n} - \frac{1}{2}\int_{-\tau}^{0} q'(s)\|f(s)\|^2_{\mathbb{C}^n}\,ds$$

$$= \left(-k+\frac{1}{2}+\frac{1}{2}q(0)\right)\|x\|^2_{\mathbb{C}^n} + \frac{1}{2}\left(\|B\|^2-q(-\tau)\right)\|f(-\tau)\|^2_{\mathbb{C}^n} + \int_{-\tau}^{0}\frac{-q'(s)}{2q(s)}q(s)\|f(s)\|^2_{\mathbb{C}^n}\,ds.$$

Note that $q(s)=\tau^{-2}\|B\|^2s^2+1$, $\forall s\in[-\tau,0]$, then we have

$$\begin{cases} q(0)=1, \quad \|B\|^2-q(-\tau)<0, \\ q'(s)<0, \quad \forall s\in[-\tau,0], \\ 0\le \frac{-q'(s)}{2q(s)} = \frac{-\tau^{-2}\|B\|^2 s}{\tau^{-2}\|B\|^2s^2+1} \le \frac{\tau^{-2}\|B\|^2|s|}{2\tau^{-1}\|B\||s|} = \frac{\|B\|}{2\tau}. \end{cases}$$

Let

$$M=\max\left\{-k+1, \frac{\|B\|}{2\tau}\right\}.$$

Then we conclude

$$\mathrm{Re}\langle \mathcal{A}X, X\rangle_1 \le M\left[\|x\|^2_{\mathbb{C}^n} + \int_{-\tau}^{0} q(s)\|f(s)\|^2_{\mathbb{C}^n}\,ds\right] = M\langle X, X\rangle_1.$$

This is the required (2.9). The proof is complete. □

Lemma 2 *Let $\mathcal{A}$ be given by (2.4) and (2.5) and let*

$$\Delta(\lambda)=\lambda+k-Be^{-\lambda\tau} = \begin{bmatrix} \lambda+k & -b_{12}e^{-\lambda\tau} & 0 & 0 & 0 & \cdots & 0 \\ 0 & \lambda+k & -b_{23}e^{-\lambda\tau} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ -ce^{-\lambda\tau} & 0 & 0 & \ddots & -b_{m,m+1}e^{-\lambda\tau} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \lambda+k & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \ddots & -b_{(n-1)n}e^{-\lambda\tau} \\ -b_{n1}e^{-\lambda\tau} & 0 & 0 & \cdots & 0 & \cdots & \lambda+k \end{bmatrix}, \tag{2.10}$$

where $\lambda\in\mathbb{C}$. If $\det\Delta(\lambda)\neq 0$, then $\lambda\in\rho(\mathcal{A})$, the resolvent set of $\mathcal{A}$. Moreover, $(\lambda-\mathcal{A})^{-1}$, the resolvent of $\mathcal{A}$, is compact and it has the following expressions:

$$\begin{cases} (\lambda-\mathcal{A})^{-1}Y = X = (x, f(s))^T \in D(\mathcal{A}), \\ \quad \forall Y=(y, g(s))^T\in\mathcal{H}, \\ x=\Delta(\lambda)^{-1}\left[y+B\int_{-\tau}^{0} e^{-\lambda(\tau+s)}g(s)\,ds\right], \\ f(s)=e^{\lambda s}x+\int_s^0 e^{\lambda(s-\xi)}g(\xi)\,d\xi. \end{cases} \tag{2.11}$$

In particular,

$$\sigma(\mathcal{A})=\left\{\lambda\in\mathbb{C} \mid \det\Delta(\lambda)=0\right\}.$$

Proof The proof is a direct computation and we omit it here. □

Theorem 1 *Let $\mathcal{A}$ be given by (2.4) and (2.5). Then $\mathcal{A}$ generates a C_0-semigroup $e^{\mathcal{A}t}$ in $\mathcal{H}$.*

Proof From Lemma 1, it has that $\mathcal{A}-M$ is dissipative in $\mathcal{H}$ and from Lemma 2, we get the right half complex plane belongs to the resolvent set of $\mathcal{A}-M$. Then, by the Lumer–Phillips theorem, $\mathcal{A}-M$ generates a C_0-semigroup of contractions $e^{(\mathcal{A}-M)t}$ in $\mathcal{H}$. Moreover, the bounded perturbation theorem of C_0-semigroups implies that $\mathcal{A}$ generates a C_0-semigroup $e^{\mathcal{A}t}$ in $\mathcal{H}$ (see [12]). The proof is complete. □

3 Spectral analysis of the system

In this section, we are going to analyze the spectrum distribution of the system operator $\mathcal{A}$. Some analytic methods in [2, 16], and [21] will be adopted here. From Lemma 2, it has that $\lambda\in\sigma(\mathcal{A})$ if and only if $\det\Delta(\lambda)=0$. So we only need to discuss the roots of $\det\Delta(\lambda)$. Note that

$$\begin{aligned}\det\Delta(\lambda)&=\det\big(\lambda+k-Be^{-\lambda\tau}\big)\\&=(\lambda+k)^n-c\gamma e^{-\lambda m\tau}(\lambda+k)^{n-m}-\alpha e^{-\lambda n\tau}\end{aligned}\tag{3.1}$$

where

$$\alpha=\prod_{j=1}^{n}b_{j,j+1},\qquad \gamma=\prod_{j=1}^{m-1}b_{j,j+1}.$$

It is clear that (3.1) has infinite number of roots which are not easy to be determined. Now we can turn to study the eigenvalues of B, and then the roots of (3.1) can be associated with the eigenvalues of B, as shown in the following lemma, which is similar to Lemma 2.1 of [1].

Lemma 3 *If λ is a root of (3.1), then there is an eigenvalue d of the matrix B for which $d=(k+\lambda)e^{\lambda\tau}$. Conversely, for any eigenvalue d of B, any solution λ of the equation $d=(k+\lambda)e^{\lambda\tau}$ will be a root of (3.1).*

Proof A root of (3.1) must satisfy

$$\det\big(\lambda+k-Be^{-\lambda\tau}\big)=0$$

or, equivalently,

$$\det\big((\lambda+k)e^{\lambda\tau}-B\big)=0.$$

Therefore, $(\lambda+k)e^{\lambda\tau}$ must equal to an eigenvalue of the matrix B. Clearly, this characterize the roots of (3.1). The proof is complete. □

Now the problem of "discussing the roots of $\det\Delta(\lambda)$" has evolved into "discussing the roots of $d-(k+\lambda)e^{\lambda\tau}=0$". From now on, for brevity, we denote

$$h(\lambda)=\lambda+k-de^{-\lambda\tau},\tag{3.2}$$

where

$$d=Re^{i\theta},\quad R\ge 0,\ 0\le\theta<2\pi,$$

denotes any eigenvalue of B. Next we will dedicated to analyze the root distribution of $h(\lambda)=\lambda+k-de^{-\lambda\tau}=0$.

We have the following lemma directly (see Corollaries 2.3 and 2.7 of [1]).

Lemma 4 *All eigenvalues d_s of B satisfy*

$$|d_s|<k,\quad \forall i=s,2,\dots,n,\tag{3.3}$$

if and only if all roots of (3.1) or (3.2) have negative real parts for all positive values of delay τ, where d_s are determined by connection weights b_{n1}, $b_{i,i+1}$, $i=1,2,\dots,n-1$ and short-cut strength c.

Lemma 5 *Let $h(\lambda)$ with $\lambda\in\mathbb{C}$ be given by (3.2). Then there are at most two real root of $h(\lambda)$ and each one is negative if existed.*

Proof If d is complex, then it follows from (3.2) that $h(\lambda)$ has no real root. If d is real, that is $d=\pm R$, then it follows from (3.2) that

$$h'(\lambda)=1+d\tau e^{-\lambda\tau}=1\pm R\tau e^{-\lambda\tau}.$$

(i) If $d=R$, then

$$h(\lambda)=\lambda+k-Re^{-\lambda\tau},\qquad h'(\lambda)=1+R\tau e^{-\lambda\tau}>0.$$

So, we have that $h(\lambda)$ is nondecreasing. Noting that

$$\lim_{\lambda\to-\infty}h(\lambda)=-\infty,\qquad \lim_{\lambda\to 0}h(\lambda)=k-R>0,$$
$$\lim_{\lambda\to+\infty}h(\lambda)=+\infty,$$

we have that $h(\lambda)$ only has one real and negative root.

(ii) If $d=-R$, then $h(\lambda)=\lambda+k+Re^{-\lambda\tau}$ and the real root if existed must be negative. Noting that

$$h'(\lambda)=1-R\tau e^{-\lambda\tau},\qquad h''(\lambda)=R\tau^2e^{-\lambda\tau}>0,$$

if $h'(\lambda)=0$, there is the unique minimum point by $\lambda_0=-\tau^{-1}\ln(1/R\tau)$. On the other hand, noting that

$$\lim_{\lambda\to-\infty} h(\lambda)=+\infty, \qquad \lim_{\lambda\to 0} h(\lambda)=k+R>0,$$
$$\lim_{\lambda\to+\infty} h(\lambda)=+\infty,$$

there are only five possible rough images for $h(\lambda)$ as Fig. 2. Thus, there are at most two real roots and each one is negative if existed. The proof is complete. □

Lemma 6 *Let $h(\lambda)$ with $\lambda\in\mathbb{C}$ be given by (3.2) and let the condition (3.3) be held. Then $h(\lambda)$ has infinitely many roots λ_n, $n\in\mathbb{N}$ in $\mathbb{C}^-$, the left half complex plane. Moreover, these roots satisfy*

$$\operatorname{Re}\lambda_n \to -\infty, \quad \text{as } n\to\infty. \tag{3.4}$$

Proof The first decision is obvious because $h(\lambda)$ is an entire function in λ and there are infinitely many roots in the complex plane. Moreover, from Lemma 4, these roots are located in the left half complex plane. Furthermore, if $|\lambda|$ large enough and $\operatorname{Re}\lambda$ bounded, then it gets

$$\big|h(\lambda)\big| \ge |\lambda| - k - |d|e^{-\tau\operatorname{Re}\lambda} > 0.$$

This yields that $\operatorname{Re}\lambda_n\to-\infty$, as $n\to\infty$. The proof is complete. □

Lemma 7 *Let $h(\lambda)$ with $\lambda\in\mathbb{C}$ be given by (3.2). Then all roots of $h(\lambda)$ are simple except one possible root λ_0 with its multiplicity two.*

Proof Firstly, from the proof of Lemma 5, we have that the possible real roots of $h(\lambda)$ are simple except one possible root λ_0 with its multiplicity two. On the other hand, if λ is a root of $h(\lambda)$ with multiplicity two, we have

$$\begin{cases} h(\lambda)=\lambda+k-de^{-\lambda\tau}=0,\\ h'(\lambda)=1+d\tau e^{-\lambda\tau}=0. \end{cases}$$

These yield

$$de^{-\lambda\tau}=-\frac{1}{\tau}, \qquad \lambda+k+\frac{1}{\tau}=0.$$

So, $\lambda=-k-\frac{1}{\tau}\in\mathbb{R}^-$, which implies that all the complex roots of $h(\lambda)$ are simple. The proof is complete. □

Now we are in a position to study the asymptotic distribution of the roots of $h(\lambda)$.

Proposition 1 *Let $h(\lambda)$ be given by (3.2). Then $h(\lambda)=\lambda+k-de^{-\lambda\tau}$, where $d=Re^{i\theta}$, has the roots given by*

$$\sigma\big(h(\lambda)\big)=\big\{\xi_n,\ \overline{\xi_n}\big\}_{n\in\mathbb{N}}\cup\{\nu_i\}, \quad i\in\mathcal{I}, \tag{3.5}$$

where ν_i is the real root of $h(\lambda)$, $\mathcal{I}\subseteq\{1,2\}$, and ξ_n has the following asymptotic expression:

$$\begin{aligned}\xi_n &= \frac{1}{\tau}\left[\ln R - \ln\frac{\theta+(2n-\frac12)\pi}{\tau}\right]\\ &\quad + i\left[\frac{\theta+(2n-\frac12)\pi}{\tau} - \frac{\ln\frac{\theta+(2n-\frac12)\pi}{\tau}}{\tau[\theta+(2n-\frac12)\pi]}\right]\\ &\quad + \mathcal{O}\big(n^{-1}\big).\end{aligned} \tag{3.6}$$

Proof The real root of $h(\lambda)$ has been discussed in Lemma 5, we write it as $\nu_i, i\in\mathcal{I}, \mathcal{I}$ is an empty set, $\{1\}$ or $\{1,2\}$. Next, since the complex roots of $h(\lambda)$ are symmetric to the real axis, we only need to find the roots of $h(\lambda)$ located on the upper complex plane.

Let $\xi=x+iy$ with $y>0$ be a root of $h(\lambda)$. Then it follows from $h(\xi)=0$ that

$$x+iy+k-Re^{i\theta}e^{-(x+iy)\tau}=0,$$

i.e.,

$$x+iy+k-Re^{-x\tau}e^{i(\theta-y\tau)}=0,$$

which gives

$$x+k-Re^{-x\tau}\cos(\theta-y\tau)=0 \tag{3.7}$$

and

$$y-Re^{-x\tau}\sin(\theta-y\tau)=0. \tag{3.8}$$

A direct computation from (3.8) yields

$$e^{x\tau}=\frac{R\sin(\theta-y\tau)}{y}. \tag{3.9}$$

Substituting this into (3.7) to get

$$x=-k+\frac{y\cos(\theta-y\tau)}{\sin(\theta-y\tau)}. \tag{3.10}$$

By (3.9), and $y>0$, we have $\sin(\theta-y\tau)>0$ that means

$$\theta-y\tau\in\big(-2n\pi,\ (-2n+1)\pi\big), \quad n\in\mathbb{N} \tag{3.11}$$

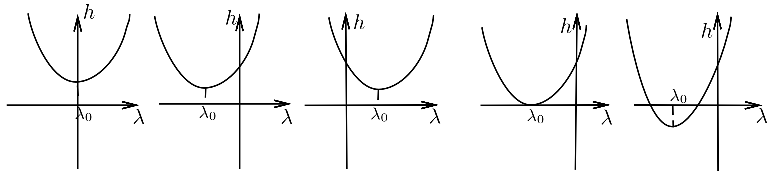

Fig. 2 The five possible images of $h(\lambda)$ when $d=-R$

and

$$y \in \left(\frac{\theta+(2n-1)\pi}{\tau}, \frac{\theta+2n\pi}{\tau}\right), \quad n \in \mathbb{N}. \tag{3.12}$$

Moreover, it follows from (3.9) that

$$x = \frac{1}{\tau}\ln\frac{R\sin(\theta-y\tau)}{y}. \tag{3.13}$$

Plugging this into the left side of (3.10) to yield

$$\ln\big(R\sin(\theta-y\tau)\big) - \ln y + k\tau - \frac{y\tau\cos(\theta-y\tau)}{\sin(\theta-y\tau)} = 0.$$

Let

$$g(y) = \ln\big(R\sin(\theta-y\tau)\big) - \ln y + k\tau - \frac{y\tau\cos(\theta-y\tau)}{\sin(\theta-y\tau)}.$$

Then we have

$$g'(y) = \frac{-\tau y\sin 2(\theta-y\tau) - \sin^2(\theta-y\tau) - \tau^2y^2}{y\sin^2(\theta-y\tau)} < 0,$$

where we have used (3.12). Since

$$\lim_{y\to\frac{\theta+(2n-1)\pi}{\tau}} g(y) = +\infty, \qquad \lim_{y\to\frac{\theta+2n\pi}{\tau}} g(y) = -\infty.$$

Hence, there exists a unique root y_n, $n\in\mathbb{N}$, on each interval

$$\left(\frac{\theta+(2n-1)\pi}{\tau}, \frac{\theta+2n\pi}{\tau}\right), \quad n\in\mathbb{N}$$

such that $g(y_n)=0$. For each $n\in\mathbb{N}$, by taking

$$x_n = \frac{1}{\tau}\ln\frac{R\sin(\theta-y_n\tau)}{y_n}, \tag{3.14}$$

then $\xi_n = x_n + iy_n$ is a root of $h(\lambda)$.

When $y_n > R$, it has $x_n < 0$, and hence,

$$y_n \to +\infty, \qquad x_n \to -\infty, \qquad \text{as } n\to+\infty. \tag{3.15}$$

Moreover, by (3.9) and (3.10), we have respectively

$$\sin(\theta-y_n\tau) = \frac{y_ne^{x_n\tau}}{R} \quad \text{and}$$

$$\sin(\theta-y_n\tau) = \frac{y_n\cos(\theta-y_n\tau)}{x_n+k}.$$

This further gives

$$\cos(\theta-y_n\tau) = \frac{1}{R}(x_n+k)e^{x_n\tau}. \tag{3.16}$$

So, due to the fact that $x_n<0$ and $x_n\to-\infty$, we have

$$\exists N, \text{ s.t. } x_n+k<0, \quad \text{if } n\ge N,$$

and $\cos(\theta-y_n\tau)<0$. This together with (3.12) further gives

$$y_n \in \left(\frac{\theta+(2n-1)\pi}{\tau}, \frac{\theta+\big(2n-\frac{1}{2}\big)\pi}{\tau}\right), \quad n\in\mathbb{N}, n\ge N. \tag{3.17}$$

Furthermore, it follows from (3.15) and (3.16) that as $n\to+\infty$,

$$(x_n+k)e^{x_n\tau}\to 0, \qquad \cos(\theta-y_n\tau)\to 0,$$
$$\theta-y_n\tau \to \left(-2n+\frac{1}{2}\right)\pi.$$

Therefore, we obtain the form of y_n by the following:

$$y_n = \frac{\theta+\big(2n-\frac{1}{2}\big)\pi+\varepsilon_n}{\tau}, \quad \varepsilon_n\in\left(-\frac{\pi}{2},0\right), \tag{3.18}$$

where $\varepsilon_n\to 0$, as $n\to+\infty$. Substituting (3.18) into $g(y_n)=0$ to get

$$0 = g(y_n) = \ln\big(R\sin(\theta-y_n\tau)\big) - \ln y_n + k\tau - \frac{y_n\tau\cos(\theta-y_n\tau)}{\sin(\theta-y_n\tau)}.$$

This gives

$$\ln R + \ln(\cos\varepsilon_n) - \ln y_n + k\tau - \frac{y_n\tau\sin\varepsilon_n}{\cos\varepsilon_n} = 0$$

and

$$\sin\varepsilon_n = \cos\varepsilon_n\left[\frac{\ln R}{y_n\tau} + \frac{\ln\cos\varepsilon_n}{y_n\tau} - \frac{\ln y_n}{y_n\tau} + \frac{k\tau}{y_n\tau}\right].$$

Expanding by the Taylor's series, we have

$$\sin\varepsilon_n = -\frac{\ln y_n}{y_n\tau} + \mathcal{O}(n^{-1}), \quad \text{as } n \to +\infty.$$

Note that $\sin\varepsilon_n = \varepsilon_n - \frac{\varepsilon_n^3}{3!} + \cdots$, we have

$$\varepsilon_n = -\frac{\ln\frac{\theta+(2n-\frac{1}{2})\pi}{\tau}}{\theta + (2n-\frac{1}{2})\pi} + \mathcal{O}(n^{-1}).$$

Hence, from (3.18), we eventually obtain the asymptotic expression of y_n by the following

$$y_n = \frac{\theta + (2n-\frac{1}{2})\pi}{\tau} - \frac{\ln\frac{\theta+(2n-\frac{1}{2})\pi}{\tau}}{\tau[\theta + (2n-\frac{1}{2})\pi]} + \mathcal{O}(n^{-1}), \tag{3.19}$$

and plugging this into (3.14) to get the asymptotic expression of x_n

$$x_n = \frac{1}{\tau}(\ln R + \ln\cos\varepsilon_n - \ln y_n) = \frac{1}{\tau}\left[\ln R - \ln\frac{\theta + (2n-\frac{1}{2})\pi}{\tau}\right] + \mathcal{O}(n^{-1}).$$

Finally, we obtain the asymptotic expression $\xi_n = x_n + iy_n$ given by (3.6). The proof is complete. □

In summary, collecting Lemmas 4–7 and Proposition 1, we can obtain the following spectrum distribution of $\mathcal{A}$ easily.

Theorem 2 *Let $\mathcal{A}$ be given by (2.4) and (2.5) and let the condition (3.3) be held. Then we have the following conclusions for the spectrum of $\mathcal{A}$:*

(1) *for each $\lambda \in \sigma(\mathcal{A})$, it has $\operatorname{Re}(\lambda) < 0$;*
(2) *$\mathcal{A}$ has infinitely many eigenvalues λ_n, $n \in \mathbb{N}$ in $\mathbb{C}^-$, and $\operatorname{Re}\lambda_n \to -\infty$, as $n \to \infty$;*
(3) *$\mathcal{A}$ has only at most $2n$ real eigenvalues;*
(4) *all the eigenvalues of $\mathcal{A}$ are simple except at most n real double eigenvalues;*
(5) *the spectrum $\sigma(\mathcal{A})$ are all the roots of $h(d_s) = \lambda + k - d_s e^{-\lambda\tau}$, where $d_s = R_s e^{i\theta_s}$, $s = 1, 2, \ldots, n$ are eigenvalues of B:*

$$\sigma(\mathcal{A}) = \bigcup_{s=1}^{n} \sigma\big(h(d_s)\big) \tag{3.20}$$

where the roots of $h(d_s)$, $s = 1, 2, \ldots, n$ are given by Proposition 1, in which d, R, and θ are replaced by d_s, R_s, and θ_s, respectively.

4 Spectrum-determined growth condition

In this section, we are going to consider the spectrum-determined growth condition for the system (2.7), which is one of the most difficult problems for infinite-dimensional systems. Our proof is based on the following characterization condition [10, Corollary 3.40] and this method has been used by the authors to treat the heat system with memory [14] and the pendulum system with position and delayed position feedbacks [15].

Lemma 8 *Let $T(t)$ be a C_0-semigroup on a Hilbert space $\mathbf{H}$ with its generator $\mathbf{A}$. Let $\omega(\mathbf{A})$ be the growth bound of $T(t)$ and*

$$s(\mathbf{A}) := \sup\{\operatorname{Re}\lambda \mid \lambda \in \sigma(\mathbf{A})\}$$

be the spectral bound of $\mathbf{A}$. Then

$$\omega(\mathbf{A}) = \inf\Big\{\omega > s(\mathbf{A}) \;\Big|\; \sup_{\tau\in\mathbb{R}}\big\|R(\sigma + i\tau, \mathbf{A})\big\| < M_\sigma < \infty, \forall\sigma \geq \omega\Big\}.$$

We also need the Lemma 1.2 of [13] (see also [6]).

Lemma 9 *Let*

$$D(\lambda) = 1 + \sum_{i=1}^{n} Q_i(\lambda)e^{\alpha_i\lambda},$$

where Q_i are polynomials of λ, α_i are some complex numbers, and n is a positive integer. Then for all λ outside those circles of radius $\varepsilon > 0$ that centered at the roots of $D(\cdot)$, one has

$$\big|D(\lambda)\big| \geq C(\varepsilon) > 0$$

for some constant $C(\varepsilon)$ that depends only on ε.

Theorem 3 *Let $\mathcal{A}$ be given by (2.4) and (2.5). Then the spectrum-determined growth condition holds true for $e^{\mathcal{A}t}$, that is, $s(\mathcal{A})=\omega(\mathcal{A})$.*

Proof By Lemma 8, the proof will be accomplished if we can show that for any $\lambda \neq 0$ and $\lambda=\alpha+i\beta$ with $\alpha \geq \omega > s(\mathcal{A})$ and $\beta\in\mathbb{R}$, there is a constant M_α such that

$$\sup_{\beta\in\mathbb{R}}\big\|R(\alpha+i\beta,\mathcal{A})\big\| \leq M_\alpha < \infty. \tag{4.1}$$

Let $\lambda=\alpha+i\beta\in\mathbb{C}$ with $\alpha\geq\omega>s(\mathcal{A})$ and $\beta\in\mathbb{R}$. Then $\lambda\in\rho(\mathcal{A})$. By Lemma 2, we have that $\forall Y=(y,g)^T\in\mathcal{H}$, there exists $X=R(\lambda,\mathcal{A})Y=(x,f)^T\in D(\mathcal{A})$ given by (2.11). For convenience, we rewrite it here:

$$\begin{cases} x=\Delta(\lambda)^{-1}\big[y+B\int_{-\tau}^{0}e^{-\lambda(\tau+s)}g(s)\,ds\big],\\ f(s)=e^{\lambda s}x+\int_{s}^{0}e^{\lambda(s-r)}g(r)\,dr.\end{cases}$$

Also by Lemma 2, we have

$$\begin{aligned} s(A)&=\sup\big\{\operatorname{Re}\lambda \mid \lambda\in\sigma(\mathcal{A})\big\}\\ &=\sup\big\{\operatorname{Re}\lambda \mid \det\Delta(\lambda)=0\big\}.\end{aligned}$$

Denote

$$\varepsilon_\alpha=\inf_{\hat{\lambda}\in\sigma(\mathcal{A}),\beta\in\mathbb{R}}\big|\hat{\lambda}-\alpha-i\beta\big|.$$

By Lemma 9, there is a positive constant $C(\varepsilon_\alpha)$ depending on α such that

$$\begin{aligned}\frac{|\det\Delta(\lambda)|}{|\lambda|^{n-1}}&=\frac{|(\lambda+k)^n-c\gamma e^{-\lambda m\tau}(\lambda+k)^m-\alpha e^{-\lambda n\tau}|}{|\lambda|^{n-1}}\\ &\geq C(\varepsilon_\alpha)>0.\end{aligned}$$

Noting that $\Delta(\lambda)^{-1}$ is a $n\times n$ matrix, after a tedious computation, we have that there exists a positive constant $\hat{M}_{1\alpha}>0$ depending on α such that

$$\sup_{\beta\in\mathbb{R}}\big\|\Delta(\lambda)^{-1}\big\|\leq\frac{\hat{M}_{1\alpha}}{C(\varepsilon_\alpha)}\doteq M_{1\alpha}<\infty.$$

Due to the estimates,

$$\begin{aligned}\int_{-\tau}^{0}e^{-\lambda(\tau+s)}e^{-\overline{\lambda}(\tau+s)}\,ds&=\int_{-\tau}^{0}e^{-2\alpha(\tau+s)}\,ds\\ &=\frac{1-e^{-2\alpha\tau}}{2\alpha},\end{aligned}$$

$$\int_{-\tau}^{0}e^{\lambda s}e^{\overline{\lambda}s}\,ds=\int_{-\tau}^{0}e^{2\alpha s}\,ds=\frac{1-e^{-2\alpha\tau}}{2\alpha},$$

$$\int_{s}^{0}e^{\lambda(s-r)}e^{\overline{\lambda}(s-r)}\,dr=\int_{s}^{0}e^{2\alpha(s-r)}\,dr=\frac{1-e^{2\alpha s}}{2\alpha}$$

and

$$\int_{-\tau}^{0}\left(\frac{1-e^{2\alpha s}}{2\alpha}\right)ds=\frac{\tau}{2\alpha}-\frac{1-e^{-2\alpha\tau}}{4\alpha^2},$$

there exist two positive constant numbers $M_{2\alpha}, M_{3\alpha}$ depending on α such that

$$\sup_{\beta\in\mathbb{R}}\int_{-\tau}^{0}\big|e^{-\lambda(\tau+s)}\big|^2\,ds\leq M_{2\alpha}<\infty,$$

$$\sup_{\beta\in\mathbb{R}}\int_{-\tau}^{0}\big|e^{\lambda s}\big|^2\,ds\leq M_{2\alpha}<\infty$$

and

$$\sup_{\beta\in\mathbb{R}}\left|\int_{-\tau}^{0}\left(\frac{1-e^{2\alpha s}}{2\alpha}\right)ds\right|\leq M_{3\alpha}<\infty.$$

Hence, we have

$$\begin{aligned}&\sup_{\beta\in\mathbb{R}}\|x\|_{\mathbb{C}^n}^2\\ &\quad=\sup_{\beta\in\mathbb{R}}\left\|\Delta(\lambda)^{-1}\left[y+B\int_{-\tau}^{0}e^{-\lambda(\tau+s)}g(s)\,ds\right]\right\|_{\mathbb{C}^n}^2\\ &\quad\leq2\Big(\sup_{\beta\in\mathbb{R}}\big\|\Delta(\lambda)^{-1}\big\|\Big)^2\bigg\{\|y\|_{\mathbb{C}^n}^2\\ &\qquad+\|B\|^2\sup_{\beta\in\mathbb{R}}\left\|\int_{-\tau}^{0}e^{-\lambda(\tau+s)}g(s)\,ds\right\|_{\mathbb{C}^n}^2\bigg\}\\ &\quad\leq2\Big(\sup_{\beta\in\mathbb{R}}\big\|\Delta(\lambda)^{-1}\big\|\Big)^2\\ &\qquad\times\bigg\{\|y\|_{\mathbb{C}^n}^2+\|B\|^2\Big(\sup_{\beta\in\mathbb{R}}\int_{-\tau}^{0}\big|e^{-\lambda(\tau+s)}\big|^2\,ds\Big)\\ &\qquad\times\Big(\int_{-\tau}^{0}\|g(s)\|_{\mathbb{C}^n}^2\,ds\Big)\bigg\}\\ &\quad\leq2M_{1\alpha}^2\|y\|_{\mathbb{C}^n}^2+2M_{1\alpha}^2b^2M_{2\alpha}\int_{-\tau}^{0}\|g(s)\|_{\mathbb{C}^n}^2\,ds\end{aligned}$$

and

$$\sup_{\beta\in\mathbb{R}}\int_{-\tau}^{0}\|f(s)\|_{\mathbb{C}^n}^2\,ds$$

$$= \sup_{\beta\in\mathbb{R}}\int_{-\tau}^{0}\left\|e^{\lambda s}x+\int_{s}^{0}e^{\lambda(s-r)}g(r)\,dr\right\|_{\mathbb{C}^n}^2 ds$$

$$\le 2\sup_{\beta\in\mathbb{R}}\int_{-\tau}^{0}\|e^{\lambda s}x\|_{\mathbb{C}^n}^2\,ds$$

$$+2\sup_{\beta\in\mathbb{R}}\int_{-\tau}^{0}\left\|\int_{s}^{0}e^{\lambda(s-r)}g(r)\,dr\right\|_{\mathbb{C}^n}^2 ds$$

$$\le 2\|x\|_{\mathbb{C}^n}^2\sup_{\beta\in\mathbb{R}}\int_{-\tau}^{0}\left|e^{\lambda s}\right|^2 ds$$

$$+2\sup_{\beta\in\mathbb{R}}\int_{-\tau}^{0}\left(\int_{s}^{0}\left|e^{\lambda(s-r)}\right|^2 dr\right)$$

$$\times\left(\int_{s}^{0}\|g(r)\|_{\mathbb{C}^n}^2\,dr\right)ds$$

$$\le 2M_{2\alpha}\|x\|_{\mathbb{C}^n}^2+2\int_{-\tau}^{0}\|g(s)\|_{\mathbb{C}^n}^2\,ds$$

$$\times\sup_{\beta\in\mathbb{R}}\int_{-\tau}^{0}\left(\frac{1-e^{2\alpha s}}{2\alpha}\right)ds$$

$$\le 2M_{2\alpha}\|x\|_{\mathbb{C}^n}^2+2M_{3\alpha}\int_{-\tau}^{0}\|g(s)\|_{\mathbb{C}^n}^2\,ds$$

$$\le 4M_{1\alpha}^2M_{2\alpha}\|y\|_{\mathbb{C}^n}^2+\left(4M_{1\alpha}^2b^2M_{2\alpha}^2+2M_{3\alpha}\right)$$

$$\times\int_{-\tau}^{0}\|g(s)\|_{\mathbb{C}^n}^2\,ds,$$

where $\|B\|=|c|+\sum_{i=1}^{n}|b_{i,i+1}|\doteq b$. Therefore, there is a positive constant $M_\alpha>0$ depending on α such that

$$\sup_{\beta\in\mathbb{R}}\|X\|_{\mathcal{H}}^2=\sup_{\beta\in\mathbb{R}}\left\{\|x\|_{\mathbb{C}^n}^2+\int_{-\tau}^{0}\|f(s)\|_{\mathbb{C}^n}^2\,ds\right\}$$

$$\le M_\alpha\left\{\|y\|_{\mathbb{C}^n}^2+\int_{-\tau}^{0}\|g(s)\|_{\mathbb{C}^n}^2\,ds\right\}$$

$$=M_\alpha\|Y\|_{\mathcal{H}}^2<\infty.$$

This yields

$$\sup_{\beta\in\mathbb{R}}\|X\|_{\mathcal{H}}\le\sqrt{M_\alpha}\|Y\|_{\mathcal{H}}<\infty,$$

so (4.1) holds. The proof is complete. □

5 Exponential stability and the effect of *c* on stability

Now, we establish the exponential stability for the system (2.7).

Theorem 4 *Let $\mathcal{A}$ be given by* (2.4) *and* (2.5) *and let the condition* (3.3) *be held. Then $e^{\mathcal{A}t}$ generated by $\mathcal{A}$ is exponentially stable, that is, there exist constants M and $\omega>0$ such that*

$$\|e^{\mathcal{A}t}\|\le Me^{-\omega t}.$$

Proof By the spectrum-determined growth condition as established by Theorem 3, the verification of the exponential stability for $e^{\mathcal{A}t}$ is determined by the spectral distribution of $\mathcal{A}$. From Theorem 2, for each $\lambda_n\in\sigma(\mathcal{A})$, we have $\operatorname{Re}\lambda_n\to-\infty$ as $n\to\infty$. Hence, $e^{\mathcal{A}t}$ is exponentially stable if and only if

$$\operatorname{Re}\lambda<0,\quad\forall\lambda\in\sigma(\mathcal{A}).$$

This has been claimed from the first decision of Theorem 2. The proof is complete. □

In what follows, we are going to establish the relation between stability and the feedback gain c, i.e., we shall find the stability interval of c. Our analysis will begin with the eigenvalues of B according to the condition (3.3).

By a simple calculation, we conclude the characteristic equation of B as follows:

$$d^n-c\gamma d^{n-m}-\alpha=0. \tag{5.1}$$

Denote $d=k\mu$, (5.1) can be written as

$$F(\mu)=\mu^n-c\gamma k^{-m}\mu^{n-m}-\alpha k^{-n}=0. \tag{5.2}$$

Furthermore, the condition (3.3) turns to

$$|\mu|<1, \tag{5.3}$$

that is, the delay-independent stability condition equal to that the roots of $F(\mu)=0$ lie inside the unit circle.

In order to verify the roots of polynomial $F(\mu)$ inside the unit circle, we shall apply Schur–Cohn criterion (see, e.g., [5] on pp. 34–36 or Proposition 5.3 of [7] on p. 27).

Proposition 2 *A necessary and sufficient condition that the polynomial*

$$F(\mu) = a_n\mu^n + a_{n-1}\mu^{n-1} + \cdots + a_1\mu + a_0, \quad a_n > 0$$

with real coefficients has all of its roots inside the unit circle is given by

$$F(1) > 0, \quad (-1)^n F(-1) > 0,$$

and the $(n-1)\times(n-1)$ *Jury matrices*

$$\Delta_{n-1}^{\pm} = \begin{pmatrix} a_n & 0 & 0 & \cdots & 0 \\ a_{n-1} & a_n & 0 & \cdots & 0 \\ a_{n-2} & a_{n-1} & a_n & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_2 & a_3 & a_4 & \cdots & a_n \end{pmatrix}$$

$$\pm \begin{pmatrix} 0 & 0 & \cdots & 0 & a_0 \\ 0 & 0 & \cdots & a_0 & a_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & a_0 & \cdots & a_{m-4} & a_{m-3} \\ a_0 & a_1 & \cdots & a_{m-3} & a_{m-2} \end{pmatrix}$$

are both positive innerwise, that is, the determinants of all of the inners of $\Delta_{n-1}^{\pm}$ *are positive. Here, the inners of a square matrix are the matrix itself and all the matrices obtained by omitting successively the first and last rows and the first and last columns.*

By using Proposition 2, we get the following necessary condition on the stability interval of c.

Theorem 5 *Suppose that all the roots of* $F(\mu)$ *lie inside the unit circle. Then*

$$1 + (-1)^{m+1}c\gamma k^{-m} + (-1)^{n+1}\alpha k^{-n} > 0 \quad \text{and}$$

$$1 - c\gamma k^{-m} - \alpha k^{-n} > 0.$$

More precisely, there are four cases:

Case 1: *if* n *is an even number and* m *is an odd number, then*

$$\begin{cases} \gamma^{-1}k^m(-1+\alpha k^{-n}) < c < \gamma^{-1}k^m(1-\alpha k^{-n}) \\ \quad \text{if } \gamma > 0, \\ \gamma^{-1}k^m(1-\alpha k^{-n}) < c < \gamma^{-1}k^m(-1+\alpha k^{-n}) \\ \quad \text{if } \gamma < 0; \end{cases}$$

Case 2: *if both* n *and* m *are even numbers, then*

$$\begin{cases} c < \gamma^{-1}k^m(1-\alpha k^{-n}) & \text{if } \gamma > 0, \\ c > \gamma^{-1}k^m(1-\alpha k^{-n}) & \text{if } \gamma < 0; \end{cases}$$

Case 3: *if both* n *and* m *are odd numbers, then*

$$\begin{cases} \gamma^{-1}k^m(-1-\alpha k^{-n}) < c < \gamma^{-1}k^m(1-\alpha k^{-n}) \\ \quad \text{if } \gamma > 0, \\ \gamma^{-1}k^m(1-\alpha k^{-n}) < c < \gamma^{-1}k^m(-1-\alpha k^{-n}) \\ \quad \text{if } \gamma < 0; \end{cases}$$

Case 4: *if* n *is an odd number and* m *is an even number, then*

$$\begin{cases} c < \min\{\gamma^{-1}k^m(1+\alpha k^{-n}), \gamma^{-1}k^m(1-\alpha k^{-n})\} \\ \quad \text{if } \gamma > 0, \\ c > \max\{\gamma^{-1}k^m(1+\alpha k^{-n}), \gamma^{-1}k^m(1-\alpha k^{-n})\} \\ \quad \text{if } \gamma < 0. \end{cases}$$

Proof By the Schur–Cohn criterion, we have

$$(-1)^n F(-1) > 0, \quad F(1) > 0.$$

Substitute this into (5.2), to get

$$1 + (-1)^{m+1}c\gamma k^{-m} + (-1)^{n+1}\alpha k^{-n} > 0 \tag{5.4}$$

and

$$1 - c\gamma k^{-m} - \alpha k^{-n} > 0 \tag{5.5}$$

which yields

$$\begin{cases} c < \gamma^{-1}k^m(1-\alpha k^{-n}) & \text{if } \gamma > 0, \\ c > \gamma^{-1}k^m(1-\alpha k^{-n}) & \text{if } \gamma < 0. \end{cases}$$

Furthermore, when n is an even number and m is an odd number, (5.4) becomes

$$1 + c\gamma k^{-m} - \alpha k^{-n} > 0,$$

which yields Case 1 directly. Other three cases can be obtained similarly and we omit their details here. □

Next, for given m and n, we present the sufficient and necessary conditions of the stability on the shortcut strength c. Denote

$$A_{n,m}^{\pm} = \gamma^{-1}k^m\big(1 \pm \alpha k^{-n}\big),$$

$$B_{n,m}^{\pm} = \pm\gamma^{-1}k^m\big(1 - \alpha^2 k^{-2n}\big).$$

Theorem 6 (i) *For $n=3, m=2$, all the roots of $F(\mu)$ lie inside the unit circle if and only if*

$$\begin{cases} B^-_{3,2} < c < \min\{B^+_{3,2}, A^-_{3,2}, A^+_{3,2}\} & \textit{if } \gamma > 0, \\ \max\{B^+_{3,2}, A^-_{3,2}, A^+_{3,2}\} < c < B^-_{3,2} & \textit{if } \gamma < 0. \end{cases}$$

(ii) *For $n=4, m=2$, all the roots of $F(\mu)$ lie inside the unit circle if and only if*

$$1 \mp \alpha k^{-4} > 0 \quad \textit{and} \quad |c| < |\gamma|^{-1} k^2 \big(1-\alpha k^{-4}\big).$$

(iii) *For $n=4, m=3$, all the roots of $F(\mu)$ lie inside the unit circle if and only if*

$$1 \mp \alpha k^{-4} > 0$$

and

$$|c| < |\gamma|^{-1} k^3 \min\big\{1-\alpha k^{-4}, \big(1-\alpha k^{-4}\big)\big(1+\alpha k^{-4}\big)^{\frac{1}{2}}, \big(1+\alpha k^{-4}\big)\big(1-\alpha k^{-4}\big)^{\frac{1}{2}}\big\}.$$

(iv) *For $n=5, m=2$, all the roots of $F(\mu)$ lie inside the unit circle if and only if*

$$1-\alpha^2 k^{-10} > 0$$

and

$$\begin{cases} \frac{\Lambda_1-\sqrt{\Lambda_2}}{2\gamma^2\alpha^2k^{-14}} < c < \min\big\{A^-_{5,2}, A^+_{5,2}, \frac{-\Lambda_1+\sqrt{\Lambda_2}}{2\gamma^2\alpha^2k^{-14}}\big\} \\ \quad \textit{if } \gamma > 0, \\ \max\big\{A^-_{5,2}, A^+_{5,2}, \frac{-\Lambda_1-\sqrt{\Lambda_2}}{2\gamma^2\alpha^2k^{-14}}\big\} \\ \quad < c < \frac{\Lambda_1+\sqrt{\Lambda_2}}{2\gamma^2\alpha^2k^{-14}} \\ \quad \textit{if } \gamma > 0, \end{cases}$$

where

$$\begin{aligned} \Lambda_1 &= \gamma k^{-2}\big(1-\alpha^2k^{-10}\big), \\ \Lambda_2 &= \gamma^2 k^{-4}\big(1-\alpha^2k^{-10}\big)^2\big(1+4\alpha^2k^{-10}\big). \end{aligned} \tag{5.6}$$

(v) *For $n=5, m=3$, all the roots of $F(\mu)$ lie inside the unit circle if and only if*

$$1-\alpha^2k^{-10} > 0,$$

and

$$\begin{cases} \max\big\{-A^+_{5,3}, \frac{\Lambda_3-\sqrt{\Lambda_4}}{2\gamma^2k^{-6}}\big\} < c < \min\big\{A^-_{5,3}, \frac{-\Lambda_3+\sqrt{\Lambda_4}}{2\gamma^2k^{-6}}\big\} \\ \quad \textit{if } \gamma > 0, \alpha > 0, \\ \max\big\{-A^+_{5,3}, \frac{-\Lambda_3-\sqrt{\Lambda_4}}{2\gamma^2k^{-6}}\big\} < c < \min\big\{A^-_{5,3}, \frac{\Lambda_3+\sqrt{\Lambda_4}}{2\gamma^2k^{-6}}\big\} \\ \quad \textit{if } \gamma > 0, \alpha < 0, \\ \max\big\{A^-_{5,3}, \frac{-\Lambda_3-\sqrt{\Lambda_4}}{2\gamma^2k^{-6}}\big\} < c < \min\big\{-A^+_{5,3}, \frac{\Lambda_3+\sqrt{\Lambda_4}}{2\gamma^2k^{-6}}\big\} \\ \quad \textit{if } \gamma < 0, \alpha > 0, \\ \max\big\{A^-_{5,3}, \frac{\Lambda_3-\sqrt{\Lambda_4}}{2\gamma^2k^{-6}}\big\} < c < \min\big\{-A^+_{5,3}, \frac{-\Lambda_3+\sqrt{\Lambda_4}}{2\gamma^2k^{-6}}\big\} \\ \quad \textit{if } \gamma < 0, \alpha < 0, \end{cases}$$

where

$$\begin{aligned} \Lambda_3 &= \gamma\alpha k^{-8}\big(1-\alpha^2k^{-10}\big), \\ \Lambda_4 &= \gamma^2k^{-6}\big(1-\alpha^2k^{-10}\big)^2\big(\alpha^2k^{-10}+4c^2\big). \end{aligned} \tag{5.7}$$

Proof (i) When $n=3, m=2$, $F(\mu)=\mu^3-c\gamma k^{-2}\mu-\alpha k^{-3}$. By Theorem 5, we have

$$\begin{cases} c < \min\{\gamma^{-1}k^2(1-\alpha k^{-3}), \gamma^{-1}k^2(1+\alpha k^{-3})\} \\ \quad \text{if } \gamma > 0, \\ c > \max\{\gamma^{-1}k^2(1-\alpha k^{-3}), \gamma^{-1}k^2(1+\alpha k^{-3})\} \\ \quad \text{if } \gamma < 0. \end{cases}$$

Noting that

$$\begin{aligned} \Delta_2^{\pm} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pm \begin{pmatrix} 0 & -\alpha k^{-3} \\ -\alpha k^{-3} & -c\gamma k^{-2} \end{pmatrix} \\ &= \begin{pmatrix} 1 & \mp\alpha k^{-3} \\ \mp\alpha k^{-3} & 1\mp c\gamma k^{-2} \end{pmatrix}, \end{aligned}$$

all the determinants of all inners are

$$\det\big(\Delta_2^{\pm}\big) = 1 \mp c\gamma k^{-2} - \alpha^2 k^{-6}.$$

So, $\det(\Delta_2^{\pm}) > 0$ yields

$$1 - c\gamma k^{-2} - \alpha^2k^{-6} > 0 \quad \text{and}$$

$$1 + c\gamma k^{-2} - \alpha^2k^{-6} > 0,$$

which deduce

$$\begin{cases} \gamma^{-1}k^2(-1+\alpha^2k^{-6}) < c < \gamma^{-1}k^2(1-\alpha^2k^{-6}) \\ \quad \text{if } \gamma > 0, \\ \gamma^{-1}k^2(1-\alpha^2k^{-6}) < c < \gamma^{-1}k^2(-1+\alpha^2k^{-6}) \\ \quad \text{if } \gamma < 0. \end{cases}$$

Thus, by Proposition 2, all the roots of $F(\mu)$ lie inside the unit circle if and only if

$$\begin{cases} \gamma^{-1}k^2(-1+\alpha^2k^{-6}) \\ \quad < c < \min\{\gamma^{-1}k^2(1-\alpha^2k^{-6}), \\ \qquad\qquad \gamma^{-1}k^2(1-\alpha k^{-3}), \\ \qquad\qquad \gamma^{-1}k^2(1+\alpha k^{-3})\} \\ \quad \text{if } \gamma > 0, \\ \max\{\gamma^{-1}k^2(1-\alpha k^{-3}), \gamma^{-1}k^2(1+\alpha k^{-3}), \\ \qquad \gamma^{-1}k^2(1-\alpha^2k^{-6})\} \\ \quad < c < \gamma^{-1}k^2(-1+\alpha^2k^{-6}) \\ \quad \text{if } \gamma < 0. \end{cases}$$

This gives claim (i).

(ii) When $n=4, m=2$, $F(\mu)=\mu^4-c\gamma k^{-2}\mu^2-\alpha k^{-4}$. By Theorem 5, we have

$$\begin{cases} c < \gamma^{-1}k^2(1-\alpha k^{-4}) & \text{if } \gamma > 0, \\ c > \gamma^{-1}k^2(1-\alpha k^{-4}) & \text{if } \gamma < 0. \end{cases}$$

Noting that

$$\Delta_3^{\pm} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -c\gamma k^{-2} & 0 & 1 \end{pmatrix} \pm \begin{pmatrix} 0 & 0 & -\alpha k^{-4} \\ 0 & -\alpha k^{-4} & 0 \\ -\alpha k^{-4} & 0 & -c\gamma k^{-2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \mp\alpha k^{-4} \\ 0 & 1\mp\alpha k^{-4} & 0 \\ -c\gamma k^{-2}\mp\alpha k^{-4} & 0 & 1\mp c\gamma k^{-2} \end{pmatrix},$$

we have all the determinants of all inners as

$$\det(\Delta_1^{\pm}) = 1\mp\alpha k^{-4},$$

$$\det(\Delta_3^{\pm}) = (1\mp\alpha k^{-4})\big[1\mp c\gamma k^{-2} - (\mp\alpha k^{-4})(-c\gamma k^{-2}\mp\alpha k^{-4})\big].$$

So, $\det(\Delta_1^{\pm}) > 0$ and $\det(\Delta_3^{\pm}) >$ yield, respectively,

$$1\mp\alpha k^{-4} > 0$$

and

$$\begin{cases} 1-c\gamma k^{-2}(1+\alpha k^{-4})-\alpha^2k^{-8} > 0, \\ 1+c\gamma k^{-2}(1+\alpha k^{-4})-\alpha^2k^{-8} > 0. \end{cases}$$

Thus, by Proposition 2, all the roots of $F(\mu)$ lie inside the unit circle if and only if

$$1\mp\alpha k^{-4} > 0, \qquad |c| < |\gamma|^{-1}k^2(1-\alpha k^{-4}).$$

Claim (ii) is then obtained.

(iii) When $n=4, m=3$, $F(\mu)=\mu^4-c\gamma k^{-3}\mu-\alpha k^{-4}$. By Theorem 5, we have

$$|c| < |\gamma|^{-1}k^3(1-\alpha k^{-4}).$$

Noting that

$$\Delta_3^{\pm} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \pm \begin{pmatrix} 0 & 0 & -\alpha k^{-4} \\ 0 & -\alpha k^{-4} & -c\gamma k^{-3} \\ -\alpha k^{-4} & -c\gamma k^{-3} & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \mp\alpha k^{-4} \\ 0 & 1\mp\alpha k^{-4} & \mp c\gamma k^{-3} \\ \mp\alpha k^{-4} & \mp c\gamma k^{-3} & 1 \end{pmatrix},$$

we have all the determinants of all inners as

$$\det(\Delta_1^{\pm}) = 1\mp\alpha k^{-4},$$
$$\det(\Delta_3^{\pm}) = 1\mp\alpha k^{-4} - c^2\gamma^2k^{-6} - \alpha^2k^{-8}(1\mp\alpha k^{-4}).$$

So, $\det(\Delta_1^{\pm}) > 0$ and $\det(\Delta_3^{\pm}) >$ yield, respectively,

$$1\mp\alpha k^{-4} > 0$$

and

$$\begin{cases} c^2\gamma^2k^{-6} < (1-\alpha k^{-4})(1-\alpha^2k^{-8}), \\ c^2\gamma^2k^{-6} < (1+\alpha k^{-4})(1-\alpha^2k^{-8}). \end{cases}$$

Thus, by Proposition 2, all the roots of $F(\mu)$ lie inside the unit circle if and only if $1\mp\alpha k^{-4} > 0$ and

$$|c| < |\gamma|^{-1}k^3\min\big\{(1-\alpha k^{-4})(1+\alpha k^{-4})^{\frac{1}{2}}, (1+\alpha k^{-4})(1-\alpha k^{-4})^{\frac{1}{2}}\big\}.$$

Claim (iii) is then proved.

(iv) When $n=5, m=2$, $F(\mu)=\mu^5-c\gamma k^{-2}\mu^3-\alpha k^{-5}$. By Theorem 5, we have

$$\begin{cases} c<\min\{\gamma^{-1}k^2(1-\alpha k^{-5}),\gamma^{-1}k^2(1+\alpha k^{-5})\} \\ \quad \text{if } \gamma>0, \\ c>\max\{\gamma^{-1}k^2(1-\alpha k^{-5}),\gamma^{-1}k^2(1+\alpha k^{-5})\} \\ \quad \text{if } \gamma<0. \end{cases}$$

Noting that

$$\Delta_4^{\pm}=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -c\gamma k^{-2} & 0 & 1 & 0 \\ 0 & -c\gamma k^{-2} & 0 & 1 \end{pmatrix} \pm \begin{pmatrix} 0 & 0 & 0 & -\alpha k^{-5} \\ 0 & 0 & -\alpha k^{-5} & 0 \\ 0 & -\alpha k^{-5} & 0 & 0 \\ -\alpha k^{-5} & 0 & 0 & -c\gamma k^{-2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \mp\alpha k^{-5} \\ 0 & 1 & \mp\alpha k^{-5} & 0 \\ -c\gamma k^{-2} & \mp\alpha k^{-5} & 1 & 0 \\ \mp\alpha k^{-5} & -c\gamma k^{-2} & 0 & 1\mp c\gamma k^{-2} \end{pmatrix},$$

we have all the determinants of all inners as

$$\det(\Delta_2^{\pm})=1-\alpha^2k^{-10},$$
$$\det(\Delta_4^{\pm})=(1\mp c\gamma k^{-2})(1-\alpha^2k^{-10}) - \alpha^2k^{-10}(1+c^2\gamma^2k^{-4}-\alpha^2k^{-10}).$$

So, $\det(\Delta_2^{\pm})>0$ and $\det(\Delta_4^{\pm})>$ yield, respectively,

$$1-\alpha^2k^{-10}>0$$

and

$$\begin{cases} (1-\alpha^2k^{-10})^2-c\gamma k^{-2}(1-\alpha^2k^{-10}) \\ \quad -c^2\gamma^2\alpha^2k^{-14}>0, \\ (1-\alpha^2k^{-10})^2+c\gamma k^{-2}(1-\alpha^2k^{-10}) \\ \quad -c^2\gamma^2\alpha^2k^{-14}>0. \end{cases}$$

Thus, by Proposition 2, all the roots of $F(\mu)$ lie inside the unit circle if and only if $1-\alpha^2k^{-10}>0$ and

$$\begin{cases} \frac{\Lambda_1-\sqrt{\Lambda_2}}{2\gamma^2\alpha^2k^{-14}}<c<\min\{A_{5,2}^-, A_{5,2}^+, \frac{-\Lambda_1+\sqrt{\Lambda_2}}{2\gamma^2\alpha^2k^{-14}}\} \\ \quad \text{if } \gamma>0, \\ \max\{A_{5,2}^-, A_{5,2}^+, \frac{-\Lambda_1-\sqrt{\Lambda_2}}{2\gamma^2\alpha^2k^{-14}}\}<c<\frac{\Lambda_1+\sqrt{\Lambda_2}}{2\gamma^2\alpha^2k^{-14}} \\ \quad \text{if } \gamma>0, \end{cases}$$

where Λ_1 and Λ_2 are given by (5.6). Claim (iv) is then concluded.

(v) When $n=5, m=3$, $F(\mu)=\mu^5-c\gamma k^{-3}\mu^2-\alpha k^{-5}$. By Theorem 5, we have

$$\begin{cases} -\gamma^{-1}k^3(1+\alpha k^{-5})<c<\gamma^{-1}k^3(1-\alpha k^{-5}) \\ \quad \text{if } \gamma>0, \\ \gamma^{-1}k^3(1-\alpha k^{-5})<c<-\gamma^{-1}k^3(1+\alpha k^{-5}) \\ \quad \text{if } \gamma<0. \end{cases}$$

Noting that

$$\Delta_4^{\pm}=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -c\gamma k^{-3} & 0 & 0 & 1 \end{pmatrix} \pm \begin{pmatrix} 0 & 0 & 0 & -\alpha k^{-5} \\ 0 & 0 & -\alpha k^{-5} & 0 \\ 0 & -\alpha k^{-5} & 0 & -c\gamma k^{-3} \\ -\alpha k^{-5} & 0 & -c\gamma k^{-3} & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \mp\alpha k^{-5} \\ 0 & 1 & \mp\alpha k^{-5} & 0 \\ 0 & \mp\alpha k^{-5} & 1 & \mp c\gamma k^{-3} \\ -c\gamma k^{-3}\mp\alpha k^{-5} & 0 & \mp c\gamma k^{-3} & 1 \end{pmatrix},$$

we have all the determinants of all inners as

$$\det(\Delta_2^{\pm})=1-\alpha^2k^{-10},$$

$$\det(\Delta_4^{\pm})=(1-\alpha^2k^{-10})^2-c^2\gamma^2k^{-6} + c\gamma k^{-3}(\mp\alpha k^{-5})(1-\alpha^2k^{-10}).$$

So, $\det(\Delta_2^{\pm})>0$ and $\det(\Delta_4^{\pm})>$ yield, respectively,

$$1-\alpha^2k^{-10}>0$$

and

$$\begin{cases} c^2\gamma^2k^{-6} + c\gamma\alpha k^{-8}(1-\alpha^2k^{-10}) \\ \quad -(1-\alpha^2k^{-10})^2 < 0, \\ c^2\gamma^2k^{-6} - c\gamma\alpha k^{-8}(1-\alpha^2k^{-10}) \\ \quad -(1-\alpha^2k^{-10})^2 < 0. \end{cases} \tag{5.8}$$

After a tedious computation, it follows from (5.8) that if $\gamma > 0, \alpha > 0$ or $\gamma < 0, \alpha < 0$,

$$\frac{\Lambda_3 - \sqrt{\Lambda_4}}{2\gamma^2k^{-6}} < c < \frac{-\Lambda_3 + \sqrt{\Lambda_4}}{2\gamma^2k^{-6}}$$

and if $\gamma < 0, \alpha > 0$ or $\gamma > 0, \alpha < 0$,

$$\frac{-\Lambda_3 - \sqrt{\Lambda_4}}{2\gamma^2k^{-6}} < c < \frac{\Lambda_3 + \sqrt{\Lambda_4}}{2\gamma^2k^{-6}},$$

where Λ_3 and Λ_4 are given by (5.7). Therefore, by Proposition 2, Claim (v) is then followed. The proof is complete. □

6 Numerical applications

In this section, we give two numerical simulations results for delayed ring neural network systems and demonstrate exactly the effect of c as a "switch" on delay-independent stability of the system.

Example 1 Consider the delayed ring neural network with four neurons described by the following:

$$\begin{cases} \dot{x_1}(t) = -5x_1(t) + 3.5f(x_2(t-\tau)), \\ \dot{x_2}(t) = -5x_2(t) + 4f(x_3(t-\tau)) \\ \qquad\qquad + cf(x_1(t-\tau)), \\ \dot{x_3}(t) = -5x_3(t) + 3f(x_4(t-\tau)), \\ \dot{x_4}(t) = -5x_4(t) + 0.5f(x_1(t-\tau)), \end{cases} \tag{6.1}$$

where

$$\begin{cases} m=2, \quad n=4, \quad k=5, \\ \tau \geq 0, \quad f(u) = \tanh(u), \\ \gamma = b_{12} = 3.5 > 0, \\ \alpha = b_{12}b_{23}b_{34}b_{41} = 21, \end{cases} \tag{6.2}$$

$$B = \begin{pmatrix} 0 & 3.5 & 0 & 0 \\ c & 0 & 4 & 0 \\ 0 & 0 & 0 & 3 \\ 0.5 & 0 & 0 & 0 \end{pmatrix}.$$

Then

$$F(\mu) = \mu^4 - \frac{3.5}{25}c\mu^2 - \frac{21}{625}.$$

By Claim (ii) of Theorem 6, all the roots of $F(\mu)$ lie inside the unit circle if and only if

$$1 \mp \alpha k^{-4} > 0 \quad \text{and} \quad |c| < |\gamma|^{-1}k^2\big(1-\alpha k^{-4}\big).$$

Noting that

$$1 \mp \alpha k^{-4} = 1 \mp \frac{21}{625} > 0 \quad \text{and}$$

$$|\gamma|^{-1}k^2\big(1-\alpha k^{-4}\big) = 6.9,$$

system (6.1) is stable for any positive delay τ if and only if $|c| < 6.9$. Therefore, the stability interval of c for (6.1) is $|c| < 6.9$.

Let $c = 1$. Matrix B given by (6.2) has four eigenvalues as

$$d_1 \approx 2.6, \qquad d_2 \approx -2.6,$$

$$d_3 \approx 1.8i, \qquad d_4 \approx -1.8i$$

Figures 3, 4, and 5 demonstrate the convergence of the state of system (6.1) for $\tau = 1$, $\tau = 3$, and $\tau = 10$, respectively.

Remark 1 From Example 1, it is found that if there is no small-world connection for (6.1), that is $c = 0$, then the system is always stable independent of time-delay $\tau > 0$.

Example 2 In Example 1, if we design a short-cut between the first and third neurons, that is $b_{31} = c$, and have the same physical parameters, then system (6.1) becomes

$$\begin{cases} \dot{x_1}(t) = -5x_1(t) + 3.5f(x_2(t-\tau)), \\ \dot{x_2}(t) = -5x_2(t) + 4f(x_3(t-\tau)), \\ \dot{x_3}(t) = -5x_3(t) + 3f(x_4(t-\tau)) \\ \qquad\qquad + cf(x_1(t-\tau)), \\ \dot{x_4}(t) = -5x_4(t) + 0.5f(x_1(t-\tau)). \end{cases} \tag{6.3}$$

By Claim (iii) of Theorem 6, we have that system (6.3) is stable for any positive delay τ if and only if $|c| < 8.63$.

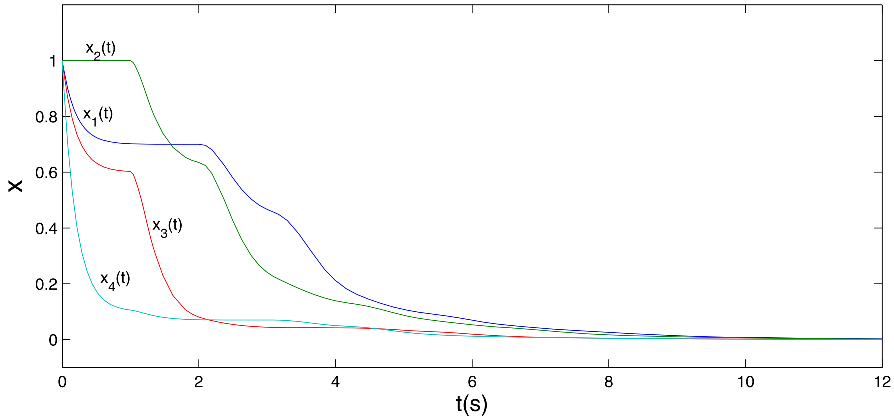

Fig. 3 The stability of the state of (6.1) when $c = 1, \tau = 1$

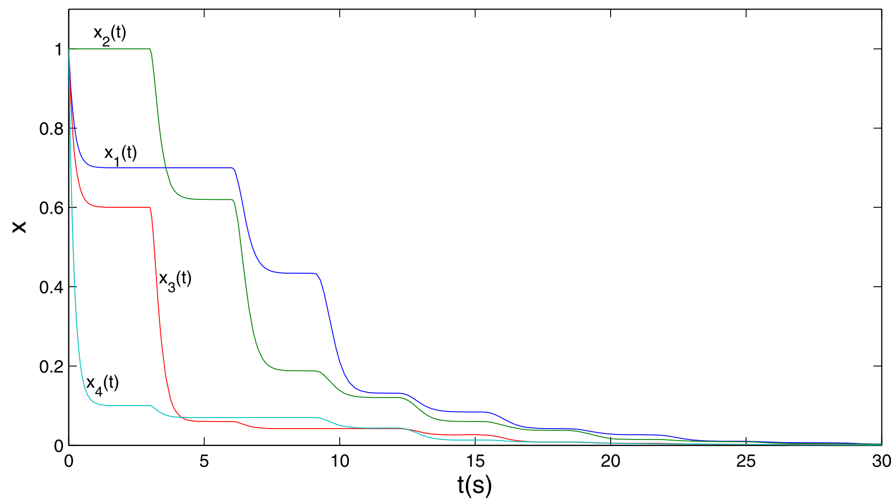

Fig. 4 The stability of the state of (6.1) when $c = 1, \tau = 3$

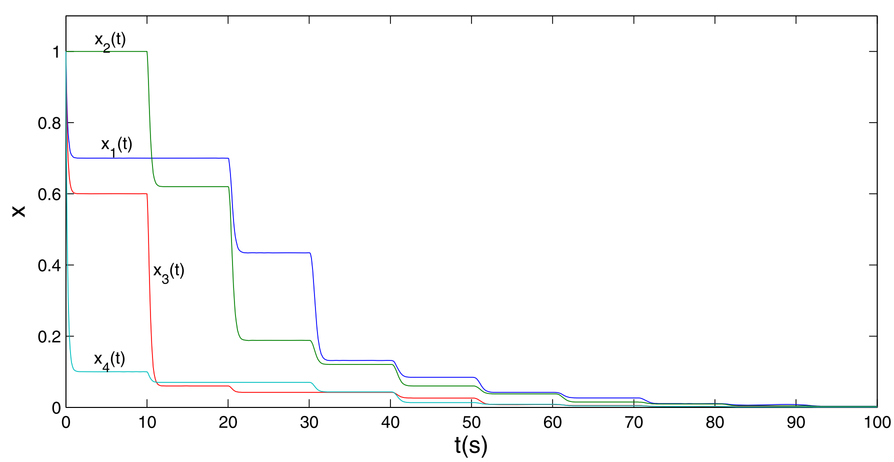

Fig. 5 The stability of the state of (6.1) when $c = 1, \tau = 10$

Remark 2 From these two examples, it is found that a small-world connection can affect stability of a system to a large degree, and the suitable short-cut c can be taken to make the system stable no matter how much time-delay is chosen. Moreover, the delay-independent stability interval of c depends continuously not only on the location of the short-cut but also the values of parameters k and $b_{i,i+1}$. On the other hand, it is found that neural networks with small-world connections are easier to be stabilized than their regular fully-connected counterparts.

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