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Exponential stability and spectral analysis of the pendulum system under position and delayed position feedbacks

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This article addresses the exponential stability and spectral analysis of the linearised pendulum system with both position and delayed position (PDP for short) feedbacks. The semigroup approach is adopted in investigation. The asymptotic spectral expression of the system is presented. It is shown that the spectrum of the system is located in the left half complex plane and its real part goes to $-\infty$ at the high frequency end when the feedback gains and the delay satisfy some additional conditions. Therefore, exponential stability of the system follows from the spectrum determined growth property. Finally, some numerical simulations of a planar pendulum are presented.

Keywords: position and delayed position feedbacks; pendulum system; spectrum; asymptotic analysis; stability

1. Introduction

It is well known that in many practical applications, a simple position feedback is not enough to get satisfactory performance for controlled systems. Additional feedbacks, such as the velocity feedback, are needed to enhance the system performance (Sontag 1990; Atay 1999). In the absence of measurement of the velocity, an observer is adopted in engineering to construct the state but this might degrade the performance to some extent (Kwakernaak and Sivan 1972). On the other hand, in the past three decades, the time-delay is introduced as a control force to stabilise the system and this has been attracted many mathematicians and engineers to focus on the study of the system with delayed inputs. Suh and Bien (1979, 1980) introduced a control design called 'proportional minus delay controller' (PMD) to improve the performance of the system based on the approximation and numerical simulation. Hu (2004) used a delayed position feedback (DPF) or a delayed velocity feedback, or both to stabilise the periodic vibration of a linear undamped oscillator. Liu and Hu (2008) presented a systematic approach for stabilising a class of linear undamped systems of multiple degrees of freedom by using both position and delayed position (PDP) feedbacks. In this article, we shall use the PDP feedback control design to stabilise the pendulum system and give a detailed and rigorous spectral analysis. The method adopted in this article can be further applied to the analysis of the system governed by the partial differential equation (PDE) with delays (Krstic 2009) and the non-collocated control systems (Guo, Wang, and Yang 2008).

One of the simplest problems in robotics is that of controlling the position of a single-link rotational joint by using a motor placed at the pivot. Mathematically, it can be considered as a planar pendulum to which an external torque is applied (Sontag 1990). The motion of the planar pendulum with an external torque can be formulated as the following second-order nonlinear differential equation (Sontag 1990; Atay 1999)

$$m\ddot{\theta}(t) + m\frac{g}{\ell}\sin\theta(t) = u(t), \tag{1.1}$$

where m is the mass, g the acceleration due to gravity, θ the angular displacement measured from the natural rest position, ℓ the length of the pendulum and $u(t)$ the value of the external torque at time t. Here, for simplicity, we assume that the friction is negligible, so that all of the mass of the rod is concentrated at the end. The stability of the two equilibria is determined by the linearised equation

$$m\ddot{y}(t) \pm m\frac{g}{\ell}y(t) = u(t), \tag{1.2}$$

where y denotes the deviation from the equilibrium, and the '+' and '−' signs correspond to the natural ($\theta = 0$) and the inverted ($\theta = \pi$) equilibrium positions, respectively. Let $k = \pm mg/\ell$. Then the pendulum system (1.2) can be rewritten as the following equation:

$$m\ddot{y}(t) + ky(t) = u(t). \tag{1.3}$$

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For a simple pendulum, it is well known that it can be stabilised in upside down position by applying an oscillating vertical force to the pivot point (Stephenson 1908; Clifford and Bishop 1998), and this has been demonstrated both experimentally, by Pippard (1987), and numerically, by Smith and Blackburn (1992) and Bartuccelli, Gentile, and Georgiou (2001). Recently, the study of the pendulum has been extended to inverted pendulums on a moving base (See Grasser, Arrigo, Colombi, and Rufer (2002), Jung and Kim (2008) Li and Zhang (2010) and the references therein).

In this article, the PDP feedbacks are designed as the following (Liu and Hu 2008):

$$u(t) = ay(t) + by(t-\tau), \tag{1.4}$$

where $a, b \in \mathbb{R}$ and τ is a positive constant representing the time-delay. Thus, the closed loop system is given by

$$m\ddot{y}(t) + (k-a)\,y(t) = by(t-\tau). \tag{1.5}$$

The aim of this article is to give a detailed spectral analysis of the system (1.5) and show that the spectrum determined growth property is true, and then to conclude the exponential stability. This article is organised as follows: in the next section (Section 2), an abstract evolution equation is set up for the system, and its well-posedness is proved. Section 3 is devoted to the detailed spectral analysis of the system. It is shown that with some conditions on the feedback gains, all eigenvalues λ_n of the system are located in the left half complex plane and their real parts $\operatorname{Re}\lambda_n$ go to $-\infty$ as $n \to \infty$. Moreover, the asymptotic spectral expression is obtained. In Section 4, the spectrum determined growth property is proved, and the exponential stability of the system is then established. In the last section (Section 5), some numerical simulations are presented to illustrate the stability of the system.

2. Abstract setting and well-posedness

In this section, we shall convert the system (1.5) into an abstract evolution equation and then discuss the well-posedness of the system. Denote

$$Z(t) = (z_1(t),\ z_2(t))^T,$$

where $z_1(t) = y(t)$, $z_2(t) = \dot{z}_1(t)$, and 'T' denotes the transpose of a vector or a matrix. Then (1.5) can be rewritten as

$$\dot{Z}(t) = A_0 Z(t) + A_1 Z(t-\tau), \tag{2.1}$$

where

$$A_0 = \begin{pmatrix} 0 & 1 \\ -\dfrac{k-a}{m} & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 \\ \dfrac{b}{m} & 0 \end{pmatrix}. \tag{2.2}$$

It is natural to set the initial data of (2.1) as the following:

$$\begin{cases} Z(0) = Z_0 = (z_{10}, z_{20}), \\ Z(s) = \phi(s), \quad s \in [-\tau, 0], \end{cases} \tag{2.3}$$

where $Z_0 \in \mathbb{C}^2$, $\phi \in L^2([-\tau,0],\mathbb{C}^2)$. We consider system (1.5) in the Hilbert state space

$$\mathcal{H} = \mathbb{C}^2 \times L^2([-\tau,0],\mathbb{C}^2)$$

equipped with the usual inner product:

$$\langle X, Y\rangle_{\mathcal{H}} = \langle x, y\rangle_{\mathbb{C}^2} + \int_{-\tau}^{0} \big\langle f(s), g(s)\big\rangle_{\mathbb{C}^2}\, \mathrm{d}s, \tag{2.4}$$

where $X = (x, f(s)) \in \mathcal{H}$, $Y = (y, g(s)) \in \mathcal{H}$. Define a linear operator $\mathcal{A} : \mathcal{H} \to \mathcal{H}$ by

$$\mathcal{A}\begin{pmatrix} x \\ f(s) \end{pmatrix} = \begin{pmatrix} A_0 & A_1\delta_1 \\ 0 & \dfrac{\mathrm{d}}{\mathrm{d}s} \end{pmatrix}\begin{pmatrix} x \\ f(s) \end{pmatrix} \tag{2.5}$$

with

$$D(\mathcal{A}) = \big\{(x, f(s)) \in \mathcal{H} \,|\, f \in H^1([-\tau,0],\mathbb{C}^2),\ f(0) = x\big\}, \tag{2.6}$$

where $\delta_1\psi = \psi(-\tau)$ and $A_0\,A_1$ are given by (2.2).

Denote

$$\begin{cases} f(t,s) = Z(t+s), \quad s \in [-\tau, 0], \\ X(t) = (Z(t), f(t,s)), \\ X(0) = X_0 := (Z_0, \phi(s)), \quad s \in [-\tau, 0], \end{cases} \tag{2.7}$$

then the system (2.1) with (2.3) can be formulated into the following abstract evolution equation $\mathcal{H}$:

$$\begin{cases} \dfrac{\mathrm{d}X(t)}{\mathrm{d}t} = \mathcal{A}X(t), \quad t > 0, \\ X(0) = X_0. \end{cases} \tag{2.8}$$

Theorem 2.1: *Let $\mathcal{A}$ be given by* (2.5) *and* (2.6) *and let*

$$\langle X, Y\rangle_1 = \langle x, y\rangle_{\mathbb{C}^2} + \int_{-\tau}^{0} q(s)\big\langle f(s), g(s)\big\rangle_{\mathbb{C}^2}\, \mathrm{d}s, \tag{2.9}$$

where $X = (x, f(s)) \in \mathcal{H}$, $Y = (y, g(s)) \in \mathcal{H}$ and $q(s) > 0$ is a bounded function in $[-\tau, 0]$. Then $\langle\cdot,\cdot\rangle_1$ is an inner product in $\mathcal{H}$ and it is equivalent to the general one given by (2.4). *Moreover, there is a suitable function $q(s)$ and a positive constant $M > 0$ such that*

$$\operatorname{Re}\langle \mathcal{A}X, X\rangle_1 \le M\langle X, X\rangle_1 \quad \forall X \in D(\mathcal{A}). \tag{2.10}$$

Hence, $\mathcal{A} - MI$ is dissipative in $\mathcal{H}$.

Proof: The first conclusion is obvious and we only need to show (2.10). For each $X=(x,f(s))\in D(\mathcal{A})$, it has

$$\langle \mathcal{A}X,X\rangle_1 = \langle A_0x + A_1f(-\tau),x\rangle_{\mathbb{C}^2} + \int_{-\tau}^{0} q(s)\left\langle \frac{\mathrm{d}}{\mathrm{d}s}f(s),f(s)\right\rangle_{\mathbb{C}^2} \mathrm{d}s.$$

A direct computation yields

$$\begin{aligned}
&\mathrm{Re}\langle \mathcal{A}X,X\rangle_1 \\
&\le \|A_0\|\|x\|_{\mathbb{C}^2}^2 + \|A_1\|\|f(-\tau)\|_{\mathbb{C}^2}\|x\|_{\mathbb{C}^2} + \frac{1}{2}\int_{-\tau}^{0} q(s)\frac{\mathrm{d}}{\mathrm{d}s}\|f(s)\|_{\mathbb{C}^2}^2\,\mathrm{d}s \\
&\le \|A_0\|\|x\|_{\mathbb{C}^2}^2 + \frac{1}{2}\left(\|A_1\|^2\|f(-\tau)\|_{\mathbb{C}^2}^2 + \|x\|_{\mathbb{C}^2}^2\right) + \frac{1}{2}\left(q(s)\|f(s)\|_{\mathbb{C}^2}^2\Big|_{-\tau}^{0} - \int_{-\tau}^{0} q'(s)\|f(s)\|_{\mathbb{C}^2}^2\,\mathrm{d}s\right) \\
&= \left(\|A_0\| + \frac{1}{2}\right)\|x\|_{\mathbb{C}^2}^2 + \frac{1}{2}\|A_1\|^2\|f(-\tau)\|_{\mathbb{C}^2}^2 + \frac{1}{2}q(0)\|f(0)\|_{\mathbb{C}^2}^2 \\
&\quad - \frac{1}{2}q(-\tau)\|f(-\tau)\|_{\mathbb{C}^2}^2 - \frac{1}{2}\int_{-\tau}^{0} q'(s)\|f(s)\|_{\mathbb{C}^2}^2\,\mathrm{d}s \\
&= \left(\|A_0\| + \frac{1}{2} + \frac{1}{2}q(0)\right)\|x\|_{\mathbb{C}^2}^2 + \frac{1}{2}\left(\|A_1\|^2 - q(-\tau)\right)\|f(-\tau)\|_{\mathbb{C}^2}^2 \\
&\quad + \int_{-\tau}^{0} \frac{-q'(s)}{2q(s)}q(s)\|f(s)\|_{\mathbb{C}^2}^2\,\mathrm{d}s.
\end{aligned}$$

If we choose

$$q(s) = \tau^{-2}\|A_1\|^2 s^2 + 1 \quad \forall s\in[-\tau,0],$$

then $q(s)\in C^1[-\tau,0]$, and $q(0)=1$,

$$\|A_1\|^2 - q(-\tau) < 0, \quad q'(s) < 0 \ \forall s\in[-\tau,0],$$

and

$$\frac{-q'(s)}{2q(s)} = \frac{-\tau^{-2}\|A_1\|^2 s}{\tau^{-2}\|A_1\|^2 s^2 + 1} \le \frac{\tau^{-2}\|A_1\|^2|s|}{2\tau^{-1}\|A_1\||s|} = \frac{\|A_1\|}{2\tau}.$$

Let

$$M = \max\left\{\|A_0\| + 1,\ \frac{\|A_1\|}{2\tau}\right\}.$$

Then, we conclude that

$$\mathrm{Re}\langle \mathcal{A}X,X\rangle_1 \le M\left[\|x\|_{\mathbb{C}^2}^2 + \int_{-\tau}^{0} q(s)\|f(s)\|_{\mathbb{C}^2}^2\,\mathrm{d}s\right] = M\langle X,X\rangle_1.$$

This is the desired (2.10). The proof is complete. □

Theorem 2.2: *Let $\mathcal{A}$ be given by* (2.5) *and* (2.6) *and let*

$$\Delta(\lambda) = \lambda I_2 - A_0 - A_1e^{-\lambda\tau} = \begin{pmatrix} \lambda & -1 \\ \dfrac{k-a}{m} - \dfrac{b}{m}e^{-\lambda\tau} & \lambda \end{pmatrix},$$

where $\lambda\in\mathbb{C}$ and I_2 is a 2×2 identity matrix. If $\det\Delta(\lambda)\neq 0$ *then $\lambda\in\rho(\mathcal{A})$, the resolvent set of $\mathcal{A}$. Moreover, $(\lambda I-\mathcal{A})^{-1}$, the resolvent of $\mathcal{A}$, is compact and it has the following expressions*:

$$\begin{cases} (\lambda I-\mathcal{A})^{-1}Y = X = (x,f(s))\in D(\mathcal{A}), \quad \forall Y=(y,g(s))\in\mathcal{H}, \\ x = \Delta(\lambda)^{-1}\left[y + A_1\displaystyle\int_{-\tau}^{0} e^{-\lambda(\tau+s)}g(s)\mathrm{d}s\right], \\ f(s) = e^{\lambda s}x + \displaystyle\int_{s}^{0} e^{\lambda(s-\xi)}g(\xi)\mathrm{d}\xi. \end{cases} \tag{2.11}$$

In particular,

$$\sigma(\mathcal{A}) = \sigma_p(\mathcal{A}) = \{\lambda\in\mathbb{C} |\ \det\Delta(\lambda)=0\}.$$

Proof: Let $\lambda\in\mathbb{C}$ and $Y=(y,g(s))\in\mathcal{H}$. The resolvent equation

$$(\lambda I-\mathcal{A})X = Y, \quad X=(x,f(s))\in D(\mathcal{A}),$$

leads to

$$\begin{cases} \lambda x - A_0x - A_1f(-\tau) = y, \\ \lambda f(s) - \dfrac{\mathrm{d}}{\mathrm{d}s}f(s) = g(s), \quad s\in[-\tau,0], \\ f(0) = x. \end{cases} \tag{2.12}$$

From the last two equations of (2.12), we get a unique solution

$$f(s) = e^{\lambda s}x + \int_{s}^{0} e^{\lambda(s-\xi)}g(\xi)\mathrm{d}\xi. \tag{2.13}$$

Substituting this into the first equation of (2.12), we have

$$\lambda x - A_0x - A_1\left(e^{-\lambda\tau}x + \int_{-\tau}^{0} e^{\lambda(-\tau-\xi)}g(\xi)\mathrm{d}\xi\right) = y$$

and

$$\Delta(\lambda)x = y + A_1\int_{-\tau}^{0} e^{-\lambda(\tau+s)}g(s)\mathrm{d}s.$$

If $\det\Delta(\lambda)\neq 0$, then

$$x = \Delta(\lambda)^{-1}\left[y + A_1\int_{-\tau}^{0} e^{-\lambda(\tau+s)}g(s)\mathrm{d}s\right], \tag{2.14}$$

and $(x,f(s))\in D(\mathcal{A})$ is uniquely determined. Hence $\lambda\in\rho(\mathcal{A})$, $\mathcal{A}$ is closed in $\mathcal{H}$, and $(\lambda I-\mathcal{A})^{-1}$ is compact. Finally, if $\det\Delta(\lambda)=0$ for $\lambda\in\mathbb{C}$, the equation $\Delta(\lambda)\eta=0$

will have a nontrivial solution $\eta \in \mathbb{C}^2$. In fact, $(\eta, e^{\lambda s}\eta) \in D(\mathcal{A})$, and

$$(\lambda I - \mathcal{A})\begin{pmatrix} \eta \\ e^{\lambda s}\eta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This implies that $\lambda \in \sigma_p(\mathcal{A})$ and

$$\sigma(A) = \sigma_p(A) = \{\lambda \in \mathbb{C} | \det \Delta(\lambda) = 0\}.$$

The proof is complete. □

Theorem 2.3: *Let $\mathcal{A}$ be given by* (2.5) *and* (2.6). *Then $\mathcal{A}$ generates a C_0-semigroup $e^{\mathcal{A}t}$ in $\mathcal{H}$.*

Proof: From Theorem 2.1, $\mathcal{A} - MI$ is dissipative in $\mathcal{H}$ and from Theorem 2.2, the right half complex plane belongs to the resolvent set of $\mathcal{A} - MI$. Then, by the Lumer–Phillips theorem, $\mathcal{A} - MI$ generates a C_0-semigroup of contractions $e^{(\mathcal{A} - MI)t}$ in $\mathcal{H}$. Moreover, the bounded perturbation theorem of C_0-semigroups implies that $\mathcal{A}$ generates a C_0-semigroup $e^{\mathcal{A}t}$ in $\mathcal{H}$ (Pazy 1983). The proof is complete. □

3. Spectral analysis of the system

In this section, we are going to analyse the spectrum distribution of the system operator $\mathcal{A}$. Some analytic methods introduced by Hagen (2005) will be adopted here. From Theorem 2.2, we have that $\lambda \in \sigma(\mathcal{A})$ if and only if $\det \Delta(\lambda) = 0$. So we only need to discuss the roots of $\det \Delta(\lambda)$. Note that

$$\begin{aligned}\det \Delta(\lambda) &= \det(\lambda I_2 - A_0 - A_1 e^{-\lambda\tau}) \\ &= \begin{vmatrix} \lambda & -1 \\ \frac{k-a}{m} - \frac{b}{m}e^{-\lambda\tau} & \lambda \end{vmatrix} = \lambda^2 + \frac{k-a}{m} - \frac{b}{m}e^{-\lambda\tau}.\end{aligned} \tag{3.1}$$

To guarantee all roots of $\det \Delta(\lambda)$ locating in the left half complex plane, we assume that the feedback gains a and b satisfy the following hypothesis:

Hypothesis: $\tau > 0$,

$$\min\left\{\frac{k-a}{m} - \frac{j^2\pi^2}{\tau^2}, \frac{(j+1)^2\pi^2}{\tau^2} - \frac{k-a}{m}\right\} > (-1)^j \frac{b}{m} > 0 \tag{3.2}$$

for some non-negative integer j.

With Hypothesis (3.2), by using Proposition 1 of Atay (1999; see also Liu and Hu (2008; Lemma 1)), we get that all roots of $\det \Delta(\lambda)$ are located in the left half complex plane. We restate this proposition as Lemma A.1 in the Appendix.

Now, we are in a position to analyse the spectrum of the system operator $\mathcal{A}$. From now on, for simplicity of notation, we denote

$$h(\lambda) = \det \Delta(\lambda) = \lambda^2 + T_0 - T_1 e^{-\lambda\tau}, \tag{3.3}$$

where

$$T_0 = \frac{k-a}{m}, \quad T_1 = \frac{b}{m}. \tag{3.4}$$

Let

$$f(\lambda) = e^{\lambda\tau} h(\lambda) = \lambda^2 e^{\lambda\tau} + T_0 e^{\lambda\tau} - T_1. \tag{3.5}$$

By Lemma A.3, we have $\mathrm{Re}\,\lambda \to -\infty$ as $\lambda \to \infty$. Thus, as $\mathrm{Re}\,\lambda \to -\infty\, f(\lambda)$ has the following asymptotic expression:

$$f(\lambda) = e^{\lambda\tau} h(\lambda) = \lambda^2 e^{\lambda\tau} - T_1 + \mathcal{O}(e^{\lambda\tau}). \tag{3.6}$$

By the Rouché's theorem, in order to consider the asymptotic distribution of the roots of $f(\lambda)$, we only need to consider the following function:

$$\tilde{f}(\lambda) = \lambda^2 e^{\lambda\tau} - T_1, \tag{3.7}$$

which can be decomposed into the following form

$$\tilde{f}(\lambda) = \begin{cases} (\lambda e^{\frac{1}{2}\lambda\tau} + \sqrt{T_1})(\lambda e^{\frac{1}{2}\lambda\tau} - \sqrt{T_1}), & \text{if } T_1 > 0, \\ (\lambda e^{\frac{1}{2}\lambda\tau} + i\sqrt{-T_1})(\lambda e^{\frac{1}{2}\lambda\tau} - i\sqrt{-T_1}), & \text{if } T_1 < 0. \end{cases} \tag{3.8}$$

Let

$$\begin{cases} f_1(\lambda) = \lambda e^{\frac{1}{2}\lambda\tau} - \sqrt{T_1}, \quad f_2(\lambda) = \lambda e^{\frac{1}{2}\lambda\tau} + \sqrt{T_1}, \\ \quad \text{for } T_1 > 0, \\ f_3(\lambda) = \lambda e^{\frac{1}{2}\lambda\tau} - i\sqrt{-T_1}, \quad f_4(\lambda) = \lambda e^{\frac{1}{2}\lambda\tau} + i\sqrt{-T_1}, \\ \quad \text{for } T_1 < 0. \end{cases} \tag{3.9}$$

Now we are going to consider the asymptotic roots of $f_i(\lambda)$, $i = 1, 2, 3, 4$, separately.

Theorem 3.1: *Let $T_1 > 0$ and let $f_1(\lambda)$ be given by* (3.9). *Then the roots of*

$$f_1(\lambda) = \lambda e^{\frac{1}{2}\lambda\tau} - \sqrt{T_1}$$

are of the form

$$\sigma(f_1(\lambda)) = \{\xi_n, \overline{\xi_n}\}_{n \in \mathbb{N}} \cup \{\nu_1\}, \tag{3.10}$$

where ν_1 is the unique real positive root of $f_1(\lambda)$, and ξ_n has the following asymptotic expression:

$$\begin{aligned}\xi_n = {} & \frac{2}{\tau}\left[\ln\sqrt{T_1} - \ln\left[\frac{(4n-1)\pi}{\tau}\right]\right] \\ & + i\left[\frac{(4n-1)\pi}{\tau} - \frac{4\ln\frac{(4n-1)\pi}{\tau}}{(4n-1)\tau\pi}\right] + \mathcal{O}(n^{-1}).\end{aligned} \tag{3.11}$$

Proof: We first look for the real root of $f_1(\lambda)$. Let $\nu \in \mathbb{R}$ be a real root of $f_1(\lambda)$. Then it follows from $T_1 > 0$ and $f_1(\nu) = 0$ that $\nu > 0$, i.e. real roots must be positive. Due to the fact that

$$f_1'(\lambda) = \left(1 + \frac{\tau}{2}\lambda\right)e^{\frac{1}{2}\tau\lambda} > 0, \quad (\text{when } \lambda > 0, \tau > 0)$$

and

$$\lim_{\lambda \to 0} f_1(\lambda) = -\sqrt{T_1} < 0, \quad \lim_{\lambda \to +\infty} f_1(\lambda) = +\infty > 0,$$

we conclude that $f_1(\lambda)$ has only one positive real root, written as ν_1.

Next, since the complex roots of $f_1(\lambda)$ are symmetric to the real axis, the roots will have the form (3.29). Hence, we only need to show that ξ_n has the asymptotic expression (3.11).

Let $\xi = x + iy$ with $y > 0$ be a root of $f_1(\lambda)$. Then it follows from $f_1(\xi) = 0$ that

$$(x + iy)e^{\frac{1}{2}x\tau}\left(\cos\frac{1}{2}y\tau + i\sin\frac{1}{2}y\tau\right) = \sqrt{T_1},$$

which gives

$$e^{\frac{1}{2}x\tau}\left(x\cos\frac{1}{2}y\tau - y\sin\frac{1}{2}y\tau\right) = \sqrt{T_1} \tag{3.12}$$

and

$$e^{\frac{1}{2}x\tau}\left(y\cos\frac{1}{2}y\tau + x\sin\frac{1}{2}y\tau\right) = 0. \tag{3.13}$$

A direct computation from (3.13) yields

$$x = -\frac{y\cos\frac{1}{2}y\tau}{\sin\frac{1}{2}y\tau}. \tag{3.14}$$

Substituting this into (3.12) leads to

$$e^{\frac{1}{2}x\tau} = -\frac{\sqrt{T_1}\sin\frac{1}{2}y\tau}{y}. \tag{3.15}$$

Due to the fact that $\sqrt{T_1} > 0$, $y > 0$ and $e^{\frac{1}{2}x\tau} > 0$, we have $-\sin\frac{1}{2}y\tau > 0$. Hence,

$$y \in \left(\frac{2(2n-1)\pi}{\tau}, \frac{4n\pi}{\tau}\right), \quad n \in \mathbb{N}. \tag{3.16}$$

Moreover, it follows from (3.15) that

$$x = \frac{2}{\tau}\left[\ln\left(-\sqrt{T_1}\sin\frac{1}{2}y\tau\right) - \ln y\right]. \tag{3.17}$$

Inserting this into the left-hand side of (3.14) yields

$$\ln\left(-\sqrt{T_1}\sin\frac{1}{2}y\tau\right) - \ln y + \frac{\tau y\cos\frac{1}{2}y\tau}{2\sin\frac{1}{2}y\tau} = 0.$$

Let

$$g(y) = \ln\left(-\sqrt{T_1}\sin\frac{1}{2}y\tau\right) - \ln y + \frac{\tau y\cos\frac{1}{2}y\tau}{2\sin\frac{1}{2}y\tau}.$$

Then

$$g'(y) = \frac{4\tau y\cos\frac{1}{2}y\tau\sin\frac{1}{2}y\tau - \tau^2y^2 - 4\sin^2\frac{1}{2}y\tau}{4y\sin^2\frac{1}{2}y\tau}$$

$$= \frac{2\tau y\sin\tau y - \tau^2y^2 - 4\sin^2\frac{1}{2}y\tau}{4y\sin^2\frac{1}{2}y\tau} < 0,$$

where we have used (3.16) and $(2\sin\tau y - \tau y) < 0$, and

$$\lim_{y \to \frac{2(2n-1)\pi}{\tau}} g(y) = +\infty, \quad \lim_{y \to \frac{4n\pi}{\tau}} g(y) = -\infty.$$

Hence, there exists a unique root y_n, $n \in \mathbb{N}$, on each interval

$$\left(\frac{2(2n-1)\pi}{\tau}, \frac{4n\pi}{\tau}\right), \quad n \in \mathbb{N}$$

such that $g(y_n) = 0$. For each $n \in \mathbb{N}$, by taking

$$x_n = \frac{2}{\tau}\ln\left(-\frac{\sqrt{T_1}\sin\frac{1}{2}y_n\tau}{y_n}\right), \tag{3.18}$$

$\xi_n = x_n + iy_n$ is a root of $f_1(\lambda)$.

When $y_n > \sqrt{T_1}$, we have

$$x_n = \frac{2}{\tau}\ln\left(-\frac{\sqrt{T_1}\sin\frac{1}{2}y_n\tau}{y_n}\right) < 0,$$

and hence,

$$y_n \to +\infty, \ x_n \to -\infty, \quad \text{as } n \to +\infty. \tag{3.19}$$

Moreover, by (3.14) and (3.15), we have, respectively,

$$\sin\frac{1}{2}y_n\tau = -\frac{y_n\cos\frac{1}{2}y_n\tau}{x_n} \quad \text{and} \quad \sin\frac{1}{2}y_n\tau = -\frac{y_n e^{\frac{1}{2}x_n\tau}}{\sqrt{T_1}}.$$

This further gives

$$x_n e^{\frac{1}{2}x_n\tau} = \sqrt{T_1}\cos\frac{1}{2}y_n\tau. \tag{3.20}$$

It follows from $x_n < 0$ and $\sqrt{T_1} > 0$ that

$$\cos\frac{1}{2}y_n\tau < 0.$$

This, together with (3.16), gives

$$y_n \in \left(\frac{2(2n-1)\pi}{\tau}, \frac{2(2n-\frac{1}{2})\pi}{\tau}\right), \quad n \in \mathbb{N}. \tag{3.21}$$

Furthermore, it follows from (3.19) and (3.20) that, as $n \to +\infty$,

$$x_n e^{\frac{1}{2}x_n\tau} \to 0, \quad \cos\frac{1}{2}y_n\tau \to 0, \quad y_n - \frac{2(2n-\frac{1}{2})\pi}{\tau} \to 0.$$

Therefore,

$$y_n = \frac{2(2n-\frac{1}{2})\pi + 2\varepsilon_n}{\tau}, \quad \varepsilon_n \in \left(-\frac{\pi}{2}, 0\right), \tag{3.22}$$

where $\varepsilon_n \to 0$, as $n \to +\infty$. Substituting (3.22) into $g(y_n)=0$, we have

$$0 = g(y_n) = \ln\sqrt{T_1} + \ln\left[-\sin\left[\left(2n-\frac{1}{2}\right)\pi + \varepsilon_n\right]\right] - \ln y_n + \frac{y_n\tau}{2}\frac{\cos\left[(2n-\frac{1}{2})\pi + \varepsilon_n\right]}{\sin\left[(2n-\frac{1}{2})\pi + \varepsilon_n\right]}.$$

This gives

$$\ln\sqrt{T_1} + \ln(\cos\varepsilon_n) - \ln y_n - \frac{y_n\tau}{2}\frac{\sin\varepsilon_n}{\cos\varepsilon_n} = 0$$

and

$$\sin\varepsilon_n = 2\cos\varepsilon_n\left[\frac{\ln\sqrt{T_1}}{y_n\tau} + \frac{\ln\cos\varepsilon_n}{y_n\tau} - \frac{\ln y_n}{y_n\tau}\right].$$

By the Taylor's series expansion,

$$\sin\varepsilon_n = -\frac{2\ln y_n}{y_n\tau} + \mathcal{O}(n^{-1}), \quad \text{as } n \to +\infty.$$

Note that $\sin\varepsilon_n = \varepsilon_n - \frac{\varepsilon_n^3}{3!} + \cdots$, we have

$$\varepsilon_n = -\frac{2\ln\frac{(4n-1)\pi}{\tau}}{(4n-1)\pi} + \mathcal{O}(n^{-1}).$$

Hence, from (3.22), we eventually obtain the asymptotic expression of y_n as the following:

$$y_n = \frac{(4n-1)\pi}{\tau} - \frac{4\ln\frac{(4n-1)\pi}{\tau}}{(4n-1)\pi\tau} + \mathcal{O}(n^{-1}). \tag{3.23}$$

Then we can get the asymptotic expression of x_n by inserting (3.23) into (3.18),

$$\begin{aligned}
x_n &= \frac{2}{\tau}\left[\ln\sqrt{T_1} + \ln\left(-\sin\frac{1}{2}y_n\tau\right) - \ln y_n\right] \\
&= \frac{2}{\tau}\left[\ln\sqrt{T_1} + \ln\left[-\sin\left[\left(2n-\frac{1}{2}\right)\pi - \frac{2\ln\frac{(4n-1)\pi}{\tau}}{(4n-1)\pi} + \mathcal{O}(n^{-1})\right]\right]\right. \\
&\quad \left. - \ln\left[\frac{(4n-1)\pi}{\tau} - \frac{4\ln\frac{(4n-1)\pi}{\tau}}{(4n-1)\pi\tau} + \mathcal{O}(n^{-1})\right]\right] \\
&= \frac{2}{\tau}\left[\ln\sqrt{T_1} + \ln\left[\cos\left[\frac{2\ln\frac{(4n-1)\pi}{\tau}}{(4n-1)\pi} + \mathcal{O}(n^{-1})\right]\right]\right. \\
&\quad \left. - \ln\left[\frac{(4n-1)\pi}{\tau} + \mathcal{O}(n^{-1})\right]\right] \\
&= \frac{2}{\tau}\left[\ln\sqrt{T_1} + \ln\left[\cos\left[\mathcal{O}\left(\frac{\ln n}{n}\right)\right]\right] - \ln\left[\frac{(4n-1)\pi}{\tau} + \mathcal{O}(n^{-1})\right]\right] \\
&= \frac{2}{\tau}\left[\ln\sqrt{T_1} - \ln\frac{(4n-1)\pi}{\tau}\right] + \mathcal{O}(n^{-1}).
\end{aligned}$$

Finally, we obtain the asymptotic expression $\xi_n = x_n + iy_n$ given by (3.11). The proof is complete. □

Since the similar arguments can be applied to derive the asymptotic roots of $f_i(\lambda)$, $i=2,3,4$, respectively, we just state the conclusions as the following three theorems and omit the tedious proofs here.

Theorem 3.2: *Let $T_1 > 0$ and let $f_2(\lambda)$ be given by (3.28). Then*

$$f_2(\lambda) = \lambda e^{\frac{1}{2}\lambda\tau} + \sqrt{T_1}$$

has the roots

$$\sigma(f_2(\lambda)) = \{\eta_n, \overline{\eta_n}\}_{n\in\mathbb{N}} \cup \{\nu_2\}, \tag{3.24}$$

where ν_2 is the unique real negative root of $f_2(\lambda)$, and η_n has the following asymptotic expression:

$$\eta_n = \frac{2}{\tau}\left[\ln\sqrt{T_1} - \ln\left[\frac{(4n+1)\pi}{\tau}\right]\right] + i\left[\frac{(4n+1)\pi}{\tau} - \frac{4\ln\frac{(4n+1)\pi}{\tau}}{(4n+1)\tau\pi}\right] + \mathcal{O}(n^{-1}). \tag{3.25}$$

Theorem 3.3: *Let $T_1 < 0$ and let $f_3(\lambda)$ be given by (3.9). Then*

$$f_3(\lambda) = \lambda e^{\frac{1}{2}\lambda\tau} - i\sqrt{-T_1}$$

has the roots

$$\sigma(f_3(\lambda)) = \{\alpha_n, \overline{\alpha_n}\}_{n\in\mathbb{N}}, \tag{3.26}$$

where α_n has the following asymptotic expression:

$$\alpha_n = \frac{2}{\tau}\left[\ln\sqrt{-T_1} - \ln\left[\frac{4n\pi}{\tau}\right]\right] + i\left[\frac{4n\pi}{\tau} - \frac{\ln\frac{4n\pi}{\tau}}{n\tau\pi}\right] + \mathcal{O}(n^{-1}). \tag{3.27}$$

Theorem 3.4: *Let $T_1 < 0$ and let $f_4(\lambda)$ be given by (3.9). Then*

$$f_4(\lambda) = \lambda e^{\frac{1}{2}\lambda\tau} + i\sqrt{-T_1}$$

has the roots

$$\sigma(f_4(\lambda)) = \{\beta_n, \overline{\beta_n}\}_{n\in\mathbb{N}}, \tag{3.28}$$

where β_n has the following asymptotic expression:

$$\beta_n = \frac{2}{\tau}\left[\ln\sqrt{-T_1} - \ln\left[\frac{(4n+2)\pi}{\tau}\right]\right] + i\left[\frac{(4n+2)\pi}{\tau} - \frac{4\ln\frac{(4n+2)\pi}{\tau}}{(4n+2)\tau\pi}\right] + \mathcal{O}(n^{-1}). \tag{3.29}$$

In summary, by Lemmas A.1–A.3 and Theorems 3.1–3.4, we can obtain the following spectrum distribution of $\mathcal{A}$.

Theorem 3.5: *Let $\mathcal{A}$ be given by* (2.5) *and* (2.6) *and let the condition* (3.2) *hold. Then we have the following conclusions for the spectrum of $\mathcal{A}$*:

(1) *for each $\lambda \in \sigma(\mathcal{A})$, it has* $\mathrm{Re}(\lambda) < 0$;
(2) *$\mathcal{A}$ has infinitely many eigenvalues λ_n, $n \in \mathbb{N}$ in $\mathbb{C}^-$, and* $\mathrm{Re}\,\lambda_n \to -\infty$, *as* $n \to \infty$;
(3) *the real eigenvalue of $\mathcal{A}$ is at most finitely many*;
(4) *the complex eigenvalues of $\mathcal{A}$ are symmetric to the real axis*;
(5) *if the feedback gain $b > 0$ in* (1.4) *or* (2.2), *i.e.* $T_1 = \frac{b}{m} > 0$, *then the condition* (3.21) *is true for some* $j = 0, 2, 4, \ldots$, *and the spectrum $\sigma(\mathcal{A})$ has the following form*:

$$\sigma(\mathcal{A}) = \{\mu_{i1}, i \in \mathcal{I}_1\} \cup \{\xi_n,\ \overline{\xi_n}\}_{n\in\mathbb{N}} \cup \{\eta_n, \overline{\eta_n}\}_{n\in\mathbb{N}}, \tag{3.30}$$

where μ_{i1} denotes the real eigenvalue of $\mathcal{A}$, $\mathcal{I}_1 \subset \{1, 2, 3\}$ an empty subset or a finite subset and ξ_n, η_n the complex eigenvalues which are of the asymptotic expressions in (3.11) *and* (3.25), *respectively*;

(6) *if the feedback gain $b < 0$ in* (1.4) *or* (2.2), *then the condition* (3.2) *is true for some* $j = 1, 3, 5, \ldots$, *and the spectrum $\sigma(\mathcal{A})$ has the following form*:

$$\sigma(\mathcal{A}) = \{\mu_{i2}, i \in \mathcal{I}_2\} \cup \{\alpha_n,\ \overline{\alpha_n}\}_{n\in\mathbb{N}} \cup \{\beta_n,\ \overline{\beta_n}\}_{n\in\mathbb{N}}, \tag{3.31}$$

where μ_{i2} denotes the real eigenvalue of $\mathcal{A}$, $\mathcal{I}_2 \subset \{1, 2\}$ an empty subset or a finite subset and α_n, β_n the complex eigenvalues which have the asymptotic expressions by (3.27) *and* (3.29), *respectively*.

Proof: By the Rouché theorem, $f(\lambda)$ given by (3.5) and $\tilde{f}(\lambda)$ given by (3.7) have the same asymptotic root distributions. Moreover, due to the fact that $\lambda \in \sigma(\mathcal{A})$ if and only if λ is a root of $h(\lambda)$ given by (3.3), and that $h(\lambda)$ and $f(\lambda)$ have the same roots, by Lemmas A.1–A.3 and Theorems 3.1–3.4, the desired results of the theorem are then obtained directly. The proof is complete. □

To end this section, we present the multiplicity of the eigenvalues of $\mathcal{A}$ and its corresponding eigenfunctions.

Theorem 3.6: *Let $\mathcal{A}$ be given by* (2.5) *and* (2.6), *and λ be an eigenvalue of $\mathcal{A}$. Assume that condition* (3.2) *holds. Then $\phi_\lambda = (x, e^{\lambda s}x)$ with $x = (1, \lambda)^T$ is an eigenfunction of $\mathcal{A}$ with respect to λ. Moreover, with the possible exception of one eigenvalue, all other eigenvalues are simple.*

Proof: Let $\lambda \in \sigma(\mathcal{A})$ and $\phi = (x, f(s)) \in D(\mathcal{A})$ be its corresponding eigenfunction. Then, $\mathcal{A}\phi = \lambda\phi$ becomes

$$\begin{cases} A_0 x + A_1 f(-\tau) = \lambda x, \\ \dfrac{\mathrm{d}}{\mathrm{d}s} f(s) = \lambda f(s), \quad s \in [-\tau, 0], \\ f(0) = x. \end{cases} \tag{3.32}$$

By the last two equations of (3.32), $f(s) = e^{\lambda s}x, s \in [-\tau, 0]$, and $\phi_\lambda = (x, e^{\lambda s}x) \in D(\mathcal{A})$. Substituting $f(s)$ into the first equation of (3.32), we get

$$A_0 x + A_1 e^{-\lambda\tau} x = \lambda x,$$

which is equivalent to solving the following algebraic equation:

$$\Delta(\lambda)x = \begin{pmatrix} \lambda & -1 \\ T_0 - T_1 e^{-\lambda\tau} & \lambda \end{pmatrix} x = 0.$$

Since λ is the root of $h(\lambda)$ given by (3.3), it is easy to see that $x = (1, \lambda)^T$ is a solution of the above algebraic equation. Hence, $\phi_\lambda = (x, e^{\lambda s}x)$ with $x = (1, \lambda)^T$ is an eigenfunction of $\mathcal{A}$ with respect to λ. Moreover, if λ_0 is a nonsimple root of $h(\lambda)$, we have $h(\lambda_0) = \lambda_0^2 + T_0 - T_1 e^{-\lambda_0\tau} = 0$. Then

$$h'(\lambda_0) = 2\lambda_0 + \tau T_1 e^{-\lambda_0\tau} = 2\lambda_0 + \tau\lambda_0^2 + \tau T_0 = 0,$$

which has the following solutions:

$$\begin{cases} \lambda_{01} = \dfrac{-1 + \sqrt{1 - \tau^2 T_0}}{\tau}, \quad \lambda_{02} = \dfrac{-1 - \sqrt{1 - \tau^2 T_0}}{\tau}, \\ \quad \text{if } \tau^2 T_0 \le 1, \\ \lambda_{01} = \dfrac{-1 + i\sqrt{\tau^2 T_0 - 1}}{\tau}, \quad \lambda_{02} = \dfrac{-1 - i\sqrt{\tau^2 T_0 - 1}}{\tau}, \\ \quad \text{if } \tau^2 T_0 > 1. \end{cases}$$

If $h(\lambda_{0i}) = 0$, $i = 1$ or 2, for some T_1, λ_{0i} is a nonsimple root and all other roots of $h(\lambda)$ are simple. Let λ be a simple root of $h(\lambda)$. Then from the above computation of the eigenfunction we know that the geometric multiplicity of λ is one. This, together with (2.11), shows that the order p of the pole of the resolvent operator of $\mathcal{A}$ is less than or equal to one. It then follows from the general formula that $m_{(a\lambda)} \le p \cdot m_{(g\lambda)} \le 1$ (Luo, Guo, and Morgul 1999; p. 148), where $m_{(a\lambda)}$, $m_{(g\lambda)}$ denote the algebraic and geometric multiplicities, respectively, that the algebraic multiplicity of λ is also one. Then λ is simple. The proof is complete. □

4. Spectrum-determined growth condition and exponential stability

Now we are in a position to consider the spectrum-determined growth property for the system (2.18), which is one of the most difficult problems for infinite-dimensional systems. Our proof is based on the following characterisation condition (Luo et al. 1999; Corollary 3.40) and this method has been used by the authors to treat the heat system with memory (Wang, Guo, and Fu 2009).

Lemma 4.1: *Let $T(t)$ be a C_0-semigroup on a Hilbert space $\mathbf{H}$ with its generator $\mathbf{A}$. Let $\omega(\mathbf{A})$ be the growth bound of $T(t)$ and*

$$s(\mathbf{A}) := \sup\{\operatorname{Re}\lambda \mid \lambda \in \sigma(\mathbf{A})\}$$

be the spectral bound of $\mathbf{A}$. Then

$$\omega(\mathbf{A}) = \inf\left\{\omega > s(\mathbf{A}) \mid \sup_{\tau\in\mathbb{R}} \|R(\sigma + i\tau, \mathbf{A})\| < M_\sigma < \infty \quad \forall\sigma \geq \omega\right\}.$$

We also need the Lemma 1.2 of Shkalikov (1986; see also Langer (1931)).

Lemma 4.2: *Let*

$$D(\lambda) = 1 + \sum_{i=1}^{n} Q_i(\lambda) e^{\alpha_i \lambda},$$

where Q_i are polynomials of λ, α_i are some complex numbers, and n is a positive integer. Then for all λ outside those circles of radius $\varepsilon > 0$ that centred at the roots of $D(\cdot)$ one has

$$|D(\lambda)| \geq C(\varepsilon) > 0$$

for some constant $C(\varepsilon)$ that depends only on ε.

Theorem 4.3: *Let $\mathcal{A}$ be given by* (2.5) *and* (2.6). *Then the spectrum-determined growth property holds true for $e^{\mathcal{A}t}$, that is, $s(\mathcal{A}) = \omega(\mathcal{A})$.*

Proof: By Lemma 4.1, the proof will be accomplished if we can show that for any $\lambda \neq 0$ and $\lambda = \alpha + i\beta$ with $\alpha \geq \omega > s(\mathcal{A})$ and $\beta \in \mathbb{R}$, there is a constant M_α such that

$$\sup_{\beta\in\mathbb{R}} \|R(\alpha + i\beta, \mathcal{A})\| \leq M_\alpha < \infty. \tag{4.1}$$

Let $\lambda = \alpha + i\beta \in \mathbb{C}$ with $\alpha \geq \omega > s(\mathcal{A})$ and $\beta \in \mathbb{R}$. Then $\lambda \in \rho(\mathcal{A})$. By Theorem 2.2, for any $Y = (y, g(s)) \in \mathcal{H}$, there exists $X = R(\lambda, \mathcal{A})Y = (x, f(s)) \in D(\mathcal{A})$ given by (2.16). For convenience, we re-state it here

$$\begin{cases} x = \Delta(\lambda)^{-1}\left[y + A_1 \displaystyle\int_{-\tau}^{0} e^{-\lambda(\tau+s)} g(s)\mathrm{d}s\right], \\ f(s) = e^{\lambda s}x + \displaystyle\int_{s}^{0} e^{\lambda(s-r)} g(r)\mathrm{d}r. \end{cases}$$

Note that $h(\lambda) = \det \Delta(\lambda) \neq 0$, and

$$\Delta(\lambda)^{-1} = \begin{pmatrix} \dfrac{\lambda}{\lambda^2 + T_0 - T_1 e^{-\lambda\tau}} & \dfrac{1}{\lambda^2 + T_0 - T_1 e^{-\lambda\tau}} \\ \dfrac{-T_0 + T_1 e^{-\lambda\tau}}{\lambda^2 + T_0 - T_1 e^{-\lambda\tau}} & \dfrac{\lambda}{\lambda^2 + T_0 - T_1 e^{-\lambda\tau}} \end{pmatrix}. \tag{4.2}$$

Since A_1 given by (2.2) and $\Delta(\lambda)^{-1}$ are 2×2 matrices, it is easy to get their norms, respectively, $\|A_1\| = |T_1|$ and

$$\begin{aligned} \|\Delta(\lambda)^{-1}\| &= \frac{2|\lambda| + 1 + |T_1 e^{-\lambda\tau} - T_0|}{|\lambda^2 + T_0 - T_1 e^{-\lambda\tau}|} \\ &\leq \frac{2|\lambda| + 1 + |T_0| + |T_1 e^{-\lambda\tau}|}{|\lambda^2 + T_0 - T_1 e^{-\lambda\tau}|} \\ &= \frac{2 + \frac{1+|T_0|}{\sqrt{\alpha^2+\beta^2}} + \frac{|T_1|e^{-\alpha\tau}}{\sqrt{\alpha^2+\beta^2}}}{|\lambda + \frac{T_0}{\lambda} - \frac{T_1 e^{-\lambda\tau}}{\lambda}|} \leq \frac{2 + \frac{1+|T_0|}{|\alpha|} + \frac{|T_1|e^{-\alpha\tau}}{|\alpha|}}{|\lambda + \frac{T_0}{\lambda} - \frac{T_1 e^{-\lambda\tau}}{\lambda}|}. \end{aligned}$$

By Theorem 2.2, it has

$$\begin{aligned} s(A) &= \sup\{\operatorname{Re}\lambda | \lambda \in \sigma(\mathcal{A})\} = \sup\{\operatorname{Re}\lambda | \lambda \in \sigma_p(\mathcal{A})\} \\ &= \sup\{\operatorname{Re}\lambda | \det \Delta(\lambda) = 0\}. \end{aligned}$$

Denote

$$\varepsilon_\alpha = \inf_{\lambda_n \in \sigma_p(\mathcal{A}),\, \beta\in\mathbb{R}} |\lambda_n - \alpha - i\beta|.$$

By Lemma 4.2, there is a positive constant $C(\varepsilon_\alpha)$ depending on α such that

$$|\det \Delta(\lambda)| = \left|\lambda + \frac{T_0}{\lambda} - \frac{T_1 e^{-\lambda\tau}}{\lambda}\right| \geq C(\varepsilon_\alpha) > 0.$$

Therefore, there exists a positive constant $M_{1\alpha} > 0$ depending on α such that

$$\sup_{\beta\in\mathbb{R}} \|\Delta(\lambda)^{-1}\| \leq M_{1\alpha} < \infty.$$

Due to the estimates

$$\int_{-\tau}^{0} e^{-\lambda(\tau+s)} e^{-\bar{\lambda}(\tau+s)} \mathrm{d}s = \int_{-\tau}^{0} e^{-2\alpha(\tau+s)} \mathrm{d}s = \frac{1 - e^{-2\alpha\tau}}{2\alpha},$$

$$\int_{-\tau}^{0} e^{\lambda s} e^{\bar{\lambda}s} \mathrm{d}s = \int_{-\tau}^{0} e^{2\alpha s} \mathrm{d}s = \frac{1 - e^{-2\alpha\tau}}{2\alpha},$$

$$\int_{s}^{0} e^{\lambda(s-r)} e^{\bar{\lambda}(s-r)} \mathrm{d}r = \int_{s}^{0} e^{2\alpha(s-r)} \mathrm{d}r = \frac{1 - e^{2\alpha s}}{2\alpha}$$

and

$$\int_{-\tau}^{0} \left(\frac{1 - e^{2\alpha s}}{2\alpha}\right) \mathrm{d}s = \frac{\tau}{2\alpha} - \frac{1 - e^{-2\alpha\tau}}{4\alpha^2},$$

there exist two positive constants $M_{2\alpha}$, $M_{3\alpha}$ depending on α such that

$$\sup_{\beta\in\mathbb{R}}\int_{-\tau}^{0}\left|e^{-\lambda(\tau+s)}\right|^2 \mathrm{d}s \le M_{2\alpha} < \infty,$$

$$\sup_{\beta\in\mathbb{R}}\int_{-\tau}^{0}\left|e^{\lambda s}\right|^2 \mathrm{d}s \le M_{2\alpha} < \infty,$$

$$\sup_{\beta\in\mathbb{R}}\left|\int_{-\tau}^{0}\left(\frac{1-e^{2\alpha s}}{2\alpha}\right)\mathrm{d}s\right| \le M_{3\alpha} < \infty.$$

Hence, we have

$$\begin{aligned}
&\sup_{\beta\in\mathbb{R}}\|x\|_{\mathbb{C}^2}^2\\
&\quad= \sup_{\beta\in\mathbb{R}}\left\|\Delta(\lambda)^{-1}\left[y + A_1\int_{-\tau}^{0}e^{-\lambda(\tau+s)}g(s)\mathrm{d}s\right]\right\|_{\mathbb{C}^2}^2\\
&\quad\le 2\left(\sup_{\beta\in\mathbb{R}}\left\|\Delta(\lambda)^{-1}\right\|\right)^2\\
&\qquad\times\left\{\|y\|_{\mathbb{C}^2}^2 + \|A_1\|^2\sup_{\beta\in\mathbb{R}}\left\|\int_{-\tau}^{0}e^{-\lambda(\tau+s)}g(s)\mathrm{d}s\right\|_{\mathbb{C}^2}^2\right\}\\
&\quad\le 2\left(\sup_{\beta\in\mathbb{R}}\|\Delta(\lambda)^{-1}\|\right)^2\\
&\qquad\times\left\{\|y\|_{\mathbb{C}^2}^2 + \|A_1\|^2\left(\sup_{\beta\in\mathbb{R}}\int_{-\tau}^{0}\left|e^{-\lambda(\tau+s)}\right|^2\mathrm{d}s\right)\right.\\
&\qquad\left.\times\left(\int_{-\tau}^{0}\|g(s)\|_{\mathbb{C}^2}^2\,\mathrm{d}s\right)\right\}\\
&\le 2M_{1\alpha}^2\|y\|_{\mathbb{C}^2}^2 + 2\left(M_{1\alpha}^2T_1^2M_{2\alpha}\right)\int_{-\tau}^{0}\|g(s)\|_{\mathbb{C}^2}^2\,\mathrm{d}s
\end{aligned}$$

and

$$\begin{aligned}
&\sup_{\beta\in\mathbb{R}}\int_{-\tau}^{0}\|f(s)\|_{\mathbb{C}^2}^2\,\mathrm{d}s\\
&=\sup_{\beta\in\mathbb{R}}\int_{-\tau}^{0}\left\|e^{\lambda s}x+\int_{s}^{0}e^{\lambda(s-r)}g(r)\mathrm{d}r\right\|_{\mathbb{C}^2}^2\mathrm{d}s\\
&\le 2\sup_{\beta\in\mathbb{R}}\int_{-\tau}^{0}\left\|e^{\lambda s}x\right\|_{\mathbb{C}^2}^2\mathrm{d}s+2\sup_{\beta\in\mathbb{R}}\int_{-\tau}^{0}\left\|\int_{s}^{0}e^{\lambda(s-r)}g(r)\mathrm{d}r\right\|_{\mathbb{C}^2}^2\mathrm{d}s\\
&\le 2\|x\|_{\mathbb{C}^2}^2\sup_{\beta\in\mathbb{R}}\int_{-\tau}^{0}\left|e^{\lambda s}\right|^2\mathrm{d}s+2\sup_{\beta\in\mathbb{R}}\int_{-\tau}^{0}\left(\int_{s}^{0}\left|e^{\lambda(s-r)}\right|^2\mathrm{d}r\right)\\
&\quad\times\left(\int_{s}^{0}\|g(r)\|_{\mathbb{C}^2}^2\,\mathrm{d}r\right)\mathrm{d}s\\
&\le 2M_{2\alpha}\|x\|_{\mathbb{C}^2}^2+2\int_{-\tau}^{0}\|g(s)\|_{\mathbb{C}^2}^2\,\mathrm{d}s\sup_{\beta\in\mathbb{R}}\int_{-\tau}^{0}\left(\frac{1-e^{2\alpha s}}{2\alpha}\right)\mathrm{d}s\\
&\le 2M_{2\alpha}\|x\|_{\mathbb{C}^2}^2+2M_{3\alpha}\int_{-\tau}^{0}\|g(s)\|_{\mathbb{C}^2}^2\,\mathrm{d}s\\
&\le 4M_{1\alpha}^2M_{2\alpha}\|y\|_{\mathbb{C}^2}^2+(4M_{1\alpha}^2T_1^2M_{2\alpha}^2+2M_{3\alpha})\int_{-\tau}^{0}\|g(s)\|_{\mathbb{C}^2}^2\,\mathrm{d}s.
\end{aligned}$$

Therefore, there is a positive constant $M_\alpha > 0$ depending on α such that

$$\begin{aligned}
\sup_{\beta\in\mathbb{R}}\|X\|_{\mathcal{H}}^2 &= \sup_{\beta\in\mathbb{R}}\left\{\|x\|_{\mathbb{C}^2}^2 + \int_{-\tau}^{0}\|f(s)\|_{\mathbb{C}^2}^2\,\mathrm{d}s\right\}\\
&\le M_\alpha\left\{\|y\|_{\mathbb{C}^2}^2 + \int_{-\tau}^{0}\|g(s)\|_{\mathbb{C}^2}^2\,\mathrm{d}s\right\}\\
&= M_\alpha\|Y\|_{\mathcal{H}}^2 < \infty.
\end{aligned}$$

This yields

$$\sup_{\beta\in\mathbb{R}}\|X\|_{\mathcal{H}} \le \sqrt{M_\alpha}\|Y\|_{\mathcal{H}} < \infty,$$

which verifies (5.3). The proof is complete. □

The following result claims the exponential stability of the system (2.13).

Theorem 4.4: *Let $\mathcal{A}$ be given by* (2.5) *and* (2.6) *and let the condition* (3.2) *hold. Then $e^{\mathcal{A}t}$ generated by $\mathcal{A}$ is exponentially stable, i.e. there exist two constants M and $\omega > 0$ such that*

$$\|e^{\mathcal{A}t}\| \le Me^{-\omega t}.$$

Proof: By the spectrum-determined growth property in Theorem 4.3, and for $\lambda_n \in \sigma(\mathcal{A})$, $\mathrm{Re}\,\lambda_n \to -\infty$ as $n\to\infty$ claimed in (2) of Theorem 3.5, $e^{\mathcal{A}t}$ is exponentially stable if and only if

$$\mathrm{Re}\,\lambda < 0, \quad \forall\lambda \in \sigma(\mathcal{A}).$$

This has been proved in (1) of Theorem 3.5. The proof is complete. □

5. Numerical applications

In this section, we present some numerical simulations for a planar pendulum with PDP feedback controls which is governed by the following equation:

$$m\ddot{y}(t) + (k-a)\,y(t) = by(t-\tau), \tag{5.3}$$

where $y(t)$ is the angular displacement, $k = \pm\frac{mg}{l}$, l the length of the pendulum, m the mass of the pendulum and g the gravitational acceleration.

Since k can change sign, for simplicity, we choose $\tau = 0.15$ s, $m = 0.5$ kg and $k = \pm 10$. When $k = 10$, taking $a = -510$, $b = -200$ and the initial value conditions $y(0) = 1$, $y'(0) = 1$. Figure 1 presents the stability convergence of the system (5.3). When $k = -10$,

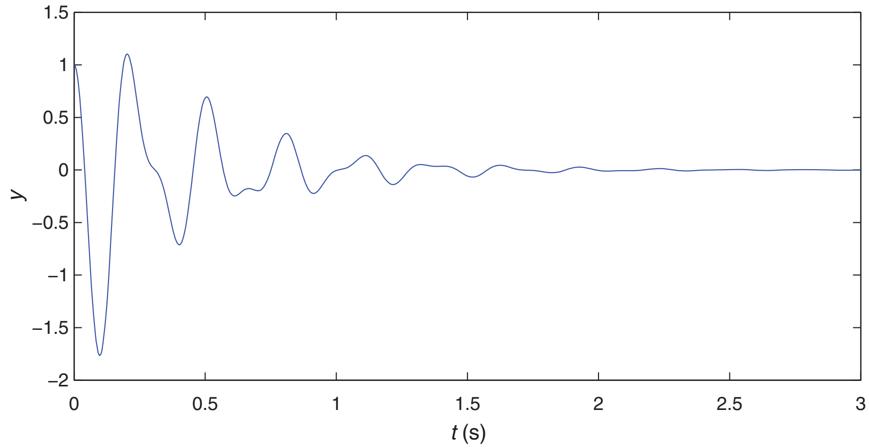

Figure 1. The stability convergence of (5.54) with $k = 10$, $a = -510$ and $b = -200$.

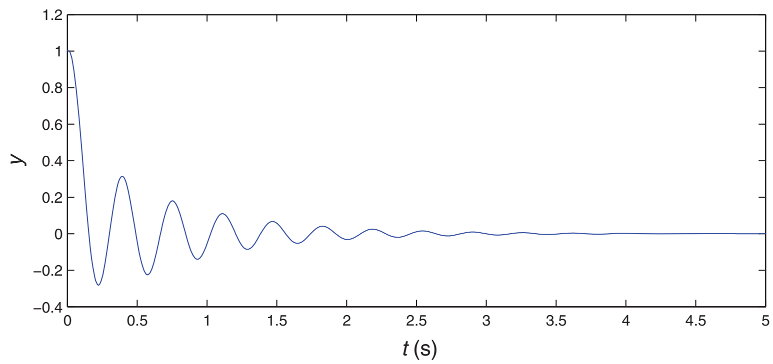

Figure 2. The stability convergence of (5.54) with $k = -10$, $a = -120$ and $b = 40$.

taking $a = -120$ and $b = 40$, Figure 2 demonstrates the stability convergence of the system (5.3).

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Appendix

This appendix is devoted to present some auxiliary results that have been used in Section 3. First, for convenience, we rewrite $\det\Delta(\lambda)$ given by (3.1) here

$$\det\Delta(\lambda) = \lambda^2 + \frac{k-a}{m} - \frac{b}{m}e^{-\lambda\tau} = \lambda^2 + T_0 - T_1 e^{-\lambda\tau}.$$

Now, we restate Proposition 1 of Atay (1999; see also Liu and Hu (2008; Lemma 1)) as Lemma A.1 here to see that all roots of $\det\Delta(\lambda)$ given by (3.1) are located in the left half complex plane.

Lemma A.1: *Let λ be the root of* $\det\Delta(\lambda)$, *i.e.* $\lambda \in \sigma(\mathcal{A})$. *Then λ has negative real part if and only if the feedback gains a and b satisfy Hypothesis* (3.2). *Moreover, the regions of roots with positive real part can be depicted in* Figure A1, *where the bold number in each region is the number of roots with positive real part.*

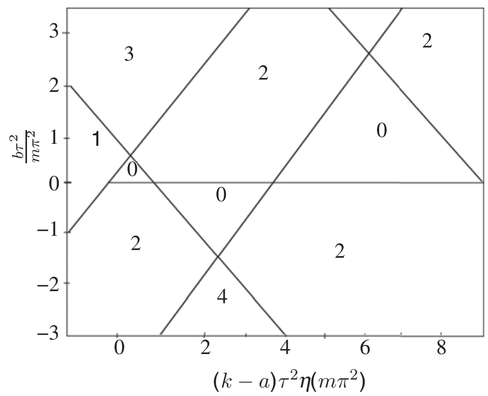

Figure A1. The regions of the roots with positive real part on the (a, b) plane. The bold number in each region denotes the number of roots with positive real part.

The real roots of det $\Delta(\lambda)$ are characterised in following lemma.

Lemma A.2: *Let $h(\lambda)$ with $\lambda \in \mathbb{C}$ be given by* (3.3) *and let the condition* (3.2) *hold. Then there is at most finitely many real roots of $h(\lambda)$ and each real root is negative if it existed. Moreover, if $T_1 > 0$, then there is at most three real roots of $h(\lambda)$ while there is at most two real roots if $T_1 < 0$.*

Proof: Let $\lambda = d$ with $d \in \mathbb{R}$. Then it follows from (3.3) that

$$h(d) = d^2 + T_0 - T_1 e^{-d\tau}.$$

Since the condition (3.2) holds, by Lemma A.1, $T_0 > 0$ and each real root is negative if it existed. Let

$$f(d) = e^{d\tau}h(d) = d^2e^{d\tau} + T_0e^{d\tau} - T_1.$$

Then

$$f'(d) = e^{d\tau}(d^2\tau + 2d + T_0\tau).$$

There are two cases:

Case I: When $T_0 \geq \tau^{-2}$, it has $f'(d) \geq 0$. So $f(d)$ is nondecreasing in d. Due to the fact that $f(d)$ is an entire function in d, there is at most one real root of $f(d)$. Moveover, if $T_1 > 0$, there is a unique negative real root of $f(d)$ while there is no real root if $T_1 \leq 0$;

Case II: When $T_0 < \tau^{-2}$, it has

$$f'(d)\begin{cases} <0, & d \in (d_1, d_2), \\ =0, & d = d_1 \text{ or } d_2, \\ >0, & \text{others}, \end{cases}$$

where

$$d_1 = \frac{-1-\sqrt{1-\tau^2T_0}}{\tau}, \quad d_2 = \frac{-1+\sqrt{1-\tau^2T_0}}{\tau}.$$

Obviously, $d_i < 0$, $i = 1, 2$. Hence the number of negative real roots of $f(d)$ is finite. The exact number of such roots will depend on the sign of T_1 and values of $f(d_i)$, $i = 1.2$.

When $T_1 > 0$, the condition (3.2) is true for any $j = 0, 2, 4, \ldots$. In this situation, $T_0 > T_1 > 0$ and

$$\lim_{d\to-\infty} f(d) = -T_1 < 0, \quad \lim_{d\to 0} f(d) = T_0 - T_1 > 0.$$

Thus,

(1) If $f(d_1)f(d_2) > 0$, $f(d)$ have only one negative real root;
(2) If $f(d_1)f(d_2) = 0$, $f(d)$ have two negative real roots;
(3) If $f(d_1)f(d_2) < 0$, $f(d)$ have three negative real roots.

When $T_1 < 0$, the condition (3.2) is true for any $j = 1, 3, 5, \ldots$. In this situation $T_0 > -T_1 > 0$ and

$$\lim_{d\to-\infty} f(d) = -T_1 > 0, \ \ f(d_1) > 0, \ \ \lim_{d\to 0} f(d) = T_0 - T_1 > 0.$$

Thus,

(1) If $f(d_2) > 0$, $f(d)$ have no real root;
(2) If $f(d_2) = 0$, $f(d)$ have only one negative real root;
(3) If $f(d_2) < 0$, $f(d)$ have two negative real roots.

Since $h(d)$ and $f(d)$ have the same root distribution, the desired conclusion follows immediately. The proof is complete. □

The following lemma shows that the real part of the high frequencies goes to $-\infty$.

Lemma A.3: *Let $h(\lambda)$ with $\lambda \in \mathbb{C}$ be given by* (3.3) *and let the condition* (3.2) *hold. Then $h(\lambda)$ has infinitely many roots λ_n, $n \in \mathbb{N}$ in $\mathbb{C}^-$, the left half complex plane. Moreover, these roots satisfy*

$$\operatorname{Re}\lambda_n \to -\infty, \quad \text{as } n \to \infty. \tag{A.1}$$

Proof: The first claim is obvious because $h(\lambda)$ is an entire function in λ and there are infinitely many roots in the complex plane. Moreover, from Lemma A.1, we infer that these roots are located in the left half complex plane. Furthermore, if $|\lambda|$ large enough and $\operatorname{Re}\lambda$ bounded, then

$$|h(\lambda)| \geq |\lambda|^2 + T_0 - |T_1| e^{-\tau \operatorname{Re}\lambda} > 0.$$

This yields that $\operatorname{Re}\lambda_n \to -\infty$, as $n \to \infty$. The proof is complete. □