Dynamic behavior of a heat equation with memory

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SUMMARY

This paper addresses the spectrum-determined growth condition for a heat equation with exponential polynomial kernel memory. By introducing some new variables, the time-variant system is transformed into a time-invariant one. The detailed spectral analysis is presented. It is shown that the system demonstrates the property of hyperbolic equation that all eigenvalues approach a line that is parallel to the imaginary axis. The residual spectral set is shown to be empty and the set of continuous spectrum is exactly characterized. The main result is the spectrum-determined growth condition that is one of the most difficult problems for infinite-dimensional systems. Consequently, a strong exponential stability result is concluded. Copyright © 2008 John Wiley & Sons, Ltd.

KEY WORDS: heat equation; spectrum; asymptotic analysis; stability

1. INTRODUCTION

It is well known that the classical one-dimensional heat equation is given by

\[ \theta_t(x, t) = \alpha \theta_{xx}(x, t) \]

where \( \theta(x, t) \) is the absolute temperature and \( \alpha > 0 \) is the thermal conductivity. It was indicated by Gurtin and Pipkin in [1] that by this equation, which is parabolic, the speed of propagation of thermal

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disturbance is infinite, and in the same paper, they proposed a general theory for heat conduction with finite propagation speed. The linearized Gurtin–Pipkin heat equation is described by

$$\theta_t(x,t) = \int_0^t k(t-s) \theta_x(x,s) \, ds$$

where the kernel $k$ is supposed to be a positive non-increasing function of its variable. Instead of the memory from infinity, Pandolfi considered in [2] this type of heat equation with memory starting from the starting point:

$$\theta_t(x,t) = \int_0^t k(t-s) \theta_x(x,s) \, ds$$

The cosine operator approach was used to study its well posedness. The earlier study on the above equation can also be found in [3, 4], where the linear theory and hyperbolicity of heat conduction for materials with memory under some assumptions were analyzed. Recently, by means of the theory of composition operators on Hardy spaces, the admissibility of observation operators and control operators for linear Volterra systems was studied in [5]. The energy decay and dynamic behavior for hyperbolic thermoelastic systems with memory type were presented in [6, 7], respectively.

In this paper, we are concerned with the same heat equation under the Dirichlet boundary condition

$$\begin{aligned}
\theta_t(x,t) - \int_0^t k(t-s) \theta_x(x,s) \, ds &= 0, \quad 0 < x < 1, \quad t > 0 \\
\theta(0, t) &= \theta(1, t) = 0 \\
\theta(x, 0) &= \theta_0(x)
\end{aligned}$$

(1)

where the kernel takes the special form of the exponential polynomial:

$$k(t) = \sum_{i=1}^N a_i^2 e^{-b_i t}, \quad a_i, b_i > 0 \in \mathbb{R}, \quad N > 1 \in \mathbb{N}$$

(2)

For simplicity, we assume that

$$b_1 < b_2 \ldots < b_N$$

(3)

Our objective is to understand the dynamic behavior of (1)–(2), particularly the large time behavior. A very special case that when the kernel is a constant

$$k(t) = x^2$$

the system (1) is reduced into the standard wave equation

$$\begin{aligned}
\theta_{tt}(x,t) - x^2 \theta_{xx}(x,t) &= 0, \quad 0 < x < 1, \quad t > 0 \\
\theta(0, t) &= \theta(1, t) = 0 \\
\theta(x, 0) &= \theta_0(x), \quad \theta_t(x, 0) = 0
\end{aligned}$$

with the wave speed $x > 0$. 

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Our study shows that for the exponential polynomial kernel, the system (1) demonstrates some typical properties of wave equation. In the next section we introduce some new variables so that the system (1) is reduced to be a time-invariant system. Section 3 is devoted to the detailed spectral analysis of the newly formulated system. It is shown that all eigenvalues approach a line that is parallel to the imaginary axis. The residual spectral set is shown to be empty and the set of continuous spectrum is exactly characterized. The main result is the spectrum-determined growth condition that is presented in Section 4. This is the big difference compared with the existing works because the system operator here is not of compact resolvent. Finally, a strongly exponential stability result is concluded.

2. TIME-INVARIANT SETUP

Introduce

\[ w_i(x, t) = a_i \int_0^t e^{-b_i(t-s)} \theta_x(x, s) \, ds, \quad i = 1, 2, \ldots, N \]  

(4)

Then, \( w_i \) satisfies

\[
\begin{cases}
(w_i)_t(x, t) = a_i \theta_x(x, t) - b_i w_i(x, t) \\
(w_i)_x(x, t) = a_i \int_0^t e^{-b_i(t-s)} \theta_{xx}(x, s) \, ds
\end{cases}
\]

(5)

and hence the first equation of (1) is reduced to

\[
\theta_t(x, t) - \sum_{i=1}^N a_i (w_i)_x(x, t) = 0
\]

(6)

Collecting (1), (5) and (6) give a newly formulated system:

\[
\begin{cases}
\theta_t(x, t) - \frac{d}{dx} \sum_{i=1}^N a_i w_i(x, t) = 0, \quad 0 < x < 1, \quad t > 0 \\
(w_i)_t(x, t) = a_i \theta_x(x, t) - b_i w_i(x, t), \quad i = 1, 2, \ldots, N \\
\theta(0, t) = \theta(1, t) = 0 \\
\theta(x, 0) = \theta_0(x), \quad w_i(x, 0) = 0, \quad i = 1, 2, \ldots, N
\end{cases}
\]

(7)

It is seen that the system (7) is a time-invariant system. Nevertheless, the system (7) is dissipative with the energy function given by

\[
E(t) = \frac{1}{2} \int_0^1 \left[ \theta^2(x, t) + \sum_{i=1}^N w_i^2(x, t) \right] \, dx
\]

(8)

Actually, a formal computation shows that

\[
\dot{E}(t) = -\sum_{i=1}^N b_i \int_0^1 w_i^2(x, t) \, dx \leq 0
\]

(9)
Motivated by the energy function (8), it is natural to consider the system (7) in the energy Hilbert space $\mathcal{H}$ is given by

$$\mathcal{H} = (L^2(0, 1))^{N+1}$$

(10)
equipped with the usual inner product. Now, define the system operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ by

$$\mathcal{A} \begin{bmatrix} \theta \\ w_1 \\ \vdots \\ w_N \end{bmatrix}^\top = \begin{bmatrix} \frac{d}{dx} \sum_{i=1}^N a_i w_i \\ a_1 \theta' - b_1 w_1 \\ \vdots \\ a_N \theta' - b_N w_N \end{bmatrix}^\top,$$

$$D(\mathcal{A}) = \begin{bmatrix} \theta \\ w_1 \\ \vdots \\ w_N \end{bmatrix}^\top, \quad \sum_{i=1}^N a_i w_i \in H^1(0, 1)$$

(11)

Then, system (7) can be formulated as an abstract evolution equation in $\mathcal{H}$

$$\frac{d}{dt} Y(t) = \mathcal{A} Y(t), \quad Y(0) = Y_0$$

(12)

where $Y(\cdot, t) = (\theta(\cdot, t), w_1(\cdot, t), \ldots, w_N(\cdot, t))$ and $Y_0 = [\theta_0, 0, \ldots, 0]$.

**Lemma 2.1**

Let $\mathcal{A}$ be defined by (11). Then $\mathcal{A}^{-1}$ exists, and hence $0 \in \rho(\mathcal{A})$, the resolvent set of $\mathcal{A}$. Moreover, $\mathcal{A}$ is dissipative and thus $\mathcal{A}$ generates a $C_0$-semigroup of contractions $e^{\mathcal{A}t}$ on $\mathcal{H}$.

**Proof**

Let $G = [g, f_1, \ldots, f_N] \in \mathcal{H}$. Solve $\mathcal{A} F = G$ for $F = [\theta, w_1, \ldots, w_N] \in D(\mathcal{A})$, that is,

$$\begin{aligned}
\frac{d}{dx} \sum_{i=1}^N a_i w_i &= g(x) \\
a_i \theta'(x) - b_i w_i(x) &= f_i(x), \quad i = 1, 2, \ldots, N \\
\theta(0) = \theta(1) &= 0
\end{aligned}$$

(13)

to give

$$w_i(x) = \frac{a_i}{b_i} \theta'(x) - \frac{1}{b_i} f_i(x), \quad \sum_{i=1}^N a_i w_i(x) = \int_0^x g(\tau) d\tau + \sum_{i=1}^N a_i w_i(0)$$

(14)

and hence

$$\sum_{i=1}^N a_i w_i(x) = \theta'(x) \sum_{i=1}^N \frac{a_i^2}{b_i} - \sum_{i=1}^N \frac{a_i}{b_i} f_i(x)$$

This together with (14) yields

$$\theta'(x) = \frac{1}{A} \int_0^x g(\tau) d\tau + \frac{1}{A} \sum_{i=1}^N a_i w_i(0) + \frac{1}{A} \sum_{i=1}^N \frac{a_i}{b_i} f_i(x), \quad A = \sum_{i=1}^N \frac{a_i^2}{b_i}$$

(15)
Using the boundary condition \( \theta(0) = 0 \) gives
\[
\theta(x) = \frac{1}{A} \int_0^x (x - \tau) g(\tau) \, d\tau + \frac{1}{A} \sum_{i=1}^N \frac{a_i}{b_i} \int_0^x f_i(\tau) \, d\tau + \frac{1}{A} \sum_{i=1}^N a_i w_i(0)x
\] (16)
and by another boundary condition \( \theta(1) = 0 \), it yields
\[
\sum_{i=1}^N a_i w_i(0) = - \int_0^1 (1 - \tau) g(\tau) \, d\tau - \sum_{i=1}^N \frac{a_i}{b_i} \int_0^1 f_i(\tau) \, d\tau
\] (17)
Collecting (14)–(17), we get the unique solution \( F \) to Equation (13). Hence, \( F \in D(\mathcal{A}) \) and \( \mathcal{A}^{-1} \) exists.

By Lumer–Phillips theorem (Theorem 4.3, [8], p. 14), the proof will be accomplished if we can show that \( \mathcal{A} \) is dissipative in \( \mathcal{H} \). Actually, for each \( F = [\theta, w_1, \ldots, w_N] \in D(\mathcal{A}) \), we have
\[
\langle \mathcal{A} F, F \rangle = \left\langle \begin{bmatrix}
\frac{d}{dx} \sum_{i=1}^N a_i w_i \\
\theta' - b_1 w_1 \\
\vdots \\
a_N \theta' - b_N w_N
\end{bmatrix}
, 
\begin{bmatrix}
\theta \\
w_1 \\
\vdots \\
w_N
\end{bmatrix}
\right\rangle
= \left\langle \frac{d}{dx} \sum_{i=1}^N a_i w_i, \theta \right\rangle_{L^2} + \sum_{i=1}^N \langle \theta', -b_i w_i \rangle_{L^2}
= \sum_{i=1}^N a_i \langle w_i, \theta' \rangle_{L^2} - \sum_{i=1}^N a_i \langle w_i, \theta' \rangle_{L^2} + \sum_{i=1}^N a_i \langle \theta', w_i \rangle_{L^2} - \sum_{i=1}^N b_i \|w_i\|_{L^2}^2
= - \sum_{i=1}^N a_i \langle w_i, \theta' \rangle_{L^2} - \sum_{i=1}^N a_i \langle \theta', w_i \rangle_{L^2} - \sum_{i=1}^N b_i \|w_i\|_{L^2}^2
which yields
\[
\text{Re} \langle \mathcal{A} F, F \rangle = - \sum_{i=1}^N b_i \|w_i\|_{L^2}^2 \leq 0
\] (18)
This is the required result. The proof is complete.

3. SPECTRAL ANALYSIS

In this section, we analyze the spectral distribution of the system (12) in the complex plane. We first investigate the point spectrum of \( \mathcal{A} \). The eigenvalue problem \( \mathcal{A} F = \lambda F \) for \( \lambda \in \mathbb{C} \) and
\[ 0 \neq F = [\theta, w_1, \ldots, w_N] \in D(\mathcal{A}) \] reads

\[
\begin{cases}
\lambda \theta(x) - \frac{d}{dx} \sum_{i=1}^{N} a_i w_i(x) = 0, & 0 < x < 1 \\
\lambda w_i(x) - a_i \theta'(x) + b_i w_i(x) = 0, & i = 1, 2, \ldots, N, \quad 0 < x < 1 \\
\theta(0) = \theta(1) = 0
\end{cases}
\]

(19)

**Lemma 3.1**
\[ \lambda = -b_i, \ i = 1, 2, \ldots, N \] are eigenvalues of \( \mathcal{A} \), which correspond to eigenfunctions \( e_{i+1}, \ i = 1, 2, \ldots, N \), respectively, where \( e_i \) is a constant function whose element is the \( i \)th element of the canonical basis of \( \mathbb{R}^{N+1} \). Moreover, each of these eigenvalues is algebraically simple.

**Proof**

We only give the proof for \( \lambda = -b_1 \) because other cases can be treated similarly. Let \( \lambda = -b_1 \) and \( F = [\theta, w_1, \ldots, w_N] \in D(\mathcal{A}) \). Solve \( (\mathcal{A} + b_1) F = 0 \), that is

\[
\begin{cases}
b_1 \theta(x) + \frac{d}{dx} \sum_{i=1}^{N} a_i w_i(x) = 0, & 0 < x < 1 \\
\lambda w_i(x) - a_i \theta'(x) + b_i w_i(x) = 0, & i = 1, 2, \ldots, N, \quad 0 < x < 1 \\
\theta(0) = \theta(1) = 0
\end{cases}
\]

(20)

to obtain \( \theta \equiv 0 \) from the second equation of (20) and boundary condition \( \theta(0) = 0 \). This in turn, together with the third equation of (20) yields

\[ (b_i - b_1) w_i(x) = 0, \quad i = 2, \ldots, N \]

Since \( b_1 \neq b_i, \ i = 2, \ldots, N \) by (3) we arrive at

\[ w_i(x) \equiv 0, \quad i = 2, \ldots, N \]

By the first equation of (20) this yields

\[ w_1'(x) = 0, \quad 0 < x < 1 \]

Therefore, \( e_2 \) is an eigenfunction of \( \mathcal{A} \) corresponding to \( -b_1 \). Further computation of \( (\mathcal{A} + b_1) F_1 = -e_2 \), where \( F_1 = [g, \phi_1, \ldots, \phi_N] \in D(\mathcal{A}) \), yields

\[
\begin{cases}
b_1 g(x) + \frac{d}{dx} \sum_{i=1}^{N} a_i \phi_i(x) = 0, & 0 < x < 1 \\
\lambda \phi_i(x) - a_i g'(x) = 0, & i = 2, 3, \ldots, N, \quad 0 < x < 1 \\
g(0) = g(1) = 0
\end{cases}
\]

(21)
We claim that (21) has no solution since $g$ must satisfy
\[ g'(x) = -\frac{1}{a_1}, \quad g(0) = g(1) = 0 \]
which is impossible. Hence, $\lambda = -b_1$ is an algebraically simple eigenvalue of $\mathscr{A}$. The proof is complete. \hfill \Box

When $\lambda \neq -b_i$, $i = 1, 2, \ldots, N$, it follows from (19) that
\[ w_i(x) = \frac{a_i \theta'(x)}{\lambda + b_i}, \quad i = 1, 2, \ldots, N \]
and $\theta$ satisfies the following equation:
\[
\begin{aligned}
\lambda \theta(x) - \theta''(x) \sum_{i=1}^{N} \frac{a_i^2}{\lambda + b_i} &= 0, \quad 0 < x < 1 \\
\theta(0) &= \theta(1) = 0
\end{aligned}
\]

**Lemma 3.2**

\[ \Lambda = \left\{ \lambda \in \mathbb{C}, \sum_{i=1}^{N} \frac{a_i^2}{\lambda + b_i} = 0 \right\} \subset \sigma_p(\mathscr{A}) \]

where $\sigma_p(\mathscr{A})$ stands for the point spectrum set of $\mathscr{A}$.

**Proof**

Obviously, $0 \notin \Lambda$. Suppose $\lambda_0 \in \Lambda$ is an eigenvalue of $\mathscr{A}$. Then, (23) becomes
\[ \theta(x) \equiv 0 \]
This together with (22) yields $w_i \equiv 0$, $i = 1, 2, \ldots, N$. This contradiction proves the required result. \hfill \Box

**Proposition 3.1**

Let $\Lambda$ be given by (24). Then there are totally $N - 1$ number of elements in $\Lambda$, and they are located separately in the following $N - 1$ different intervals:
\[ (-b_N, -b_{N-1}), (-b_{N-1}, -b_{N-2}), \ldots, (-b_2, -b_1) \]

**Proof**

Since $-b_i \notin \Lambda$ for any $i = 1, 2, \ldots, N$, $\sum_{i=1}^{N} \frac{a_i^2}{\lambda + b_i} = 0$ is equivalent to $q(\lambda) = 0$, where
\[ q(\lambda) := \sum_{i=1}^{N} a_i^2 \prod_{j=1, j \neq i}^{N} (\lambda + b_j) \]

Thus, the elements of $\Lambda$ are zeros of $q(\lambda)$. However, $q(\lambda)$ is an $(N - 1)$th order polynomial, and hence there are at most $N - 1$ number of zeros for $q(\lambda)$. Now, we find all these zeros as follows.
Notice that for any \( i = 1, 2, \ldots, N - 1 \), when \( i \) is even, \( q(-b_i) < 0, q(-b_{i+1}) > 0 \). By Rolle’s theorem, there exists a solution \( q(\lambda) = 0 \) in \((-b_{i+1}, -b_i)\). This completes the proof by (3).

\( \square \)

**Lemma 3.3**

Suppose that \( \lambda \neq 0 \) and \( \lambda \notin \Lambda \). Then, it has

\[
\lambda \prod_{i=1}^{N} (\lambda + b_i) = a(\lambda^2 + b\lambda + c)q(\lambda) + h(\lambda)
\]

where \( q(\lambda) \) is given by (26)

\[
a = \left( \sum_{i=1}^{N} a_i^2 \right)^{-1}, \quad b = a \sum_{i=1}^{N} a_i^2 b_i, \quad c = a \sum_{i=1}^{N} a_i^2 b_i^2 + b^2
\]

and \( h(\lambda) \) is a residual polynomial in \( \lambda \) with order \( N - 2 \).

**Proof**

Let

\[
\text{LHS} = \lambda \prod_{i=1}^{N} (\lambda + b_i) = a(\lambda^2 + b\lambda + c)q(\lambda) + h(\lambda) = \text{RHS}
\]

We need to show that \( a, b \) and \( c \) have the expressions given by (28). Expand LHS to give

\[
\text{LHS} = \lambda^{N+1} + \left[ \sum_{i=1}^{N} b_i \right] \lambda^{N} + \left[ \sum_{1 \leq i < j \leq N} b_i b_j \right] \lambda^{N-1} + h_1(\lambda)
\]

where \( h_1(\lambda) \) is the polynomial in \( \lambda \) with order \( N - 2 \). Similarly,

\[
\text{RHS} = \left[ a \sum_{i=1}^{N} a_i^2 \right] \lambda^{N+1} + \left[ ab \sum_{i=1}^{N} a_i^2 + a \sum_{i=1}^{N} a_i^2 \sum_{j=1, j \neq i}^{N} b_j \right] \lambda^{N}
\]

\[
+ \left[ ac \sum_{i=1}^{N} a_i^2 + ab \sum_{i=1}^{N} a_i^2 \sum_{j=1, j \neq i}^{N} b_j + a \sum_{i=1}^{N} a_i^2 \sum_{1 \leq j < s \leq N, j \neq i}^{N} b_j b_s \right] \lambda^{N-1}
\]

\[
+ h_2(\lambda)
\]

Setting \( \text{LHS} = \text{RHS} \) and comparing the coefficients of \( \lambda^{N+1}, \lambda^{N} \) and \( \lambda^{N-1} \), respectively, yield

\[
a = \left( \sum_{i=1}^{N} a_i^2 \right)^{-1}, \quad b = \sum_{i=1}^{N} b_i - a \sum_{i=1}^{N} a_i^2 b_i
\]

and

\[
c = \sum_{1 \leq i < j \leq N} b_i b_j - ab \sum_{i=1}^{N} a_i^2 \sum_{j=1, j \neq i}^{N} b_j - a \sum_{i=1}^{N} a_i^2 \sum_{1 \leq j < s \leq N, j \neq i}^{N} b_j b_s
\]
Since
\[ a \sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} a_i^2 b_j = a \sum_{i=1}^{N} \left[ \sum_{j=1}^{N} a_i^2 b_j - a_i^2 b_i \right] = a \sum_{i=1}^{N} \sum_{j=1}^{N} a_i^2 b_j - a \sum_{i=1}^{N} a_i^2 b_i \]
\[ = a \sum_{j=1}^{N} b_j \sum_{i=1}^{N} a_i^2 - a \sum_{i=1}^{N} a_i^2 b_i = \sum_{j=1}^{N} b_j - a \sum_{i=1}^{N} a_i^2 b_i \]
insert the above expression into (31) to get \( b \) in (28). Furthermore, a direct computation gives
\[ ab \sum_{i=1}^{N} a_i^2 \sum_{j=1, j \neq i}^{N} b_j = ab \sum_{i=1}^{N} a_i^2 \left[ \sum_{j=1}^{N} b_j - b_i \right] = ab \sum_{i=1}^{N} a_i^2 \sum_{j=1}^{N} b_j - ab \sum_{i=1}^{N} a_i^2 b_i \]
and
\[ a \sum_{i=1}^{N} a_i^2 \sum_{1 \leq j < s \leq N, s, j \neq i}^{N} b_j b_s \]
\[ = a \sum_{i=1}^{N} a_i^2 \left[ \sum_{1 \leq j < s \leq N} b_j b_s - \sum_{i+1}^{N} b_i b_s - \sum_{j=1}^{i-1} b_j b_i \right] \]
\[ = a \sum_{i=1}^{N} a_i^2 \left[ \sum_{1 \leq j < s \leq N} b_j b_s - \sum_{j=1}^{N} b_j b_j - b_i^2 \right] \]
\[ = a \sum_{i=1}^{N} a_i^2 \left[ \sum_{1 \leq j < s \leq N} b_j b_s - a \sum_{j=1}^{N} a_i^2 \sum_{j=1}^{N} b_i b_j - a \sum_{i=1}^{N} a_i^2 b_i \right] \]
\[ = a \sum_{1 \leq j < s \leq N} b_j b_s - \sum_{j=1}^{N} b_j - a \sum_{i=1}^{N} a_i^2 b_i \]
Plugging the above two equalities into (32), we eventually get the simplified form of \( c \) that is given by (28). The proof is complete. \( \square \)

Now, from Lemmas 3.2 and 3.3, it is seen that the eigenvalue problem (23) is equivalent to the following problem:
\[
\begin{align*}
\theta''(x) - a\theta(x)(\lambda^2 + b\lambda + c + a^{-1}h(\lambda)q^{-1}(\lambda)) &= 0, & 0 < x < 1 \\
\theta(0) &= \theta(1) = 0
\end{align*}
\tag{33}
\]
where $a$, $b$ and $c$ are constants given by (28), $q(\lambda)$ is a polynomial in $\lambda$ with order $N - 1$ given by (26) and $h(\lambda)$ is given by (27):

$$h(\lambda) = \lambda \prod_{i=1}^{N} (\lambda + b_i) - a(\lambda^2 + b\lambda + c)q(\lambda)$$

Therefore,

$$\begin{cases}
q(\lambda) \neq 0 & \text{for any } \lambda \neq 0 \text{ and } \lambda \notin \Lambda \\
h(\lambda)q^{-1}(\lambda) = O(\lambda^{-1}) & \text{as } |\lambda| \to \infty
\end{cases} \quad (34)$$

**Lemma 3.4**

Suppose $\lambda \neq 0$ and $\lambda \notin \Lambda$. Then, for $x \in [0, 1]$,

$$e^{\sqrt{\alpha}lx}, \quad e^{-\sqrt{\alpha}lx}$$

are linearly independent fundamental solutions of $\theta''(x) - a\lambda^2\theta(x) = 0$, and

$$\theta''(x) - a\theta(x)(\lambda^2 + b\lambda + c + a^{-1}h(\lambda)q^{-1}(\lambda)) = 0 \quad \text{as } |\lambda| \to \infty$$

has the following asymptotic fundamental solutions:

$$\begin{cases}
\theta_1(x) = e^{\sqrt{\alpha}lx}[\theta_{10}(x) + \theta_{11}(x)\lambda^{-1}] + O(\lambda^{-2}) \\
\theta_2(x) = e^{-\sqrt{\alpha}lx}[\theta_{20}(x) + \theta_{21}(x)\lambda^{-1}] + O(\lambda^{-2})
\end{cases} \quad (36)$$

where

$$\begin{cases}
\theta_{10}(x) = e^{(1/2)\sqrt{ab}x}, & \theta_{11}(x) = -\frac{1}{8}\sqrt{a(b^2 - 4c)}xe^{(1/2)\sqrt{ab}x} \\
\theta_{20}(x) = e^{(-1/2)\sqrt{ab}x}, & \theta_{21}(x) = \frac{1}{8}\sqrt{a(b^2 - 4c)}xe^{(-1/2)\sqrt{ab}x}
\end{cases} \quad (37)$$

**Proof**

The first claim is trivial. We only need to show that (36) is the asymptotic fundamental solutions of $\theta''(x) - a\theta(x)(\lambda^2 + b\lambda + c + a^{-1}h(\lambda)q^{-1}(\lambda)) = 0$. This can be done along the same way of Birkhoff [9] and Naimark [10]. Here, we present briefly a simple calculation to (36).

Let

$$\tilde{\theta}_1(x, \lambda) := e^{\sqrt{\alpha}lx}\left[\theta_{10}(x) + \frac{\theta_{11}(x)}{\lambda}\right], \quad \tilde{\theta}_2(x, \lambda) := e^{-\sqrt{\alpha}lx}\left[\theta_{20}(x) + \frac{\theta_{21}(x)}{\lambda}\right] \quad (38)$$

where $\theta_{10}(x)$ and $\theta_{11}(x)$ are some functions to be determined, and

$$D(\theta) = \theta''(x) - a\theta(x)(\lambda^2 + b\lambda + c + a^{-1}h(\lambda)q^{-1}(\lambda)) \quad (39)$$

where $h(\lambda)q^{-1}(\lambda) = O(\lambda^{-1})$. Substitute $\tilde{\theta}_1(x, \lambda)$ and $\tilde{\theta}_2(x, \lambda)$ into $D(\theta)$, respectively, to yield

$$e^{-\sqrt{\alpha}lx}D(\tilde{\theta}_1(x, \lambda)) = a\lambda^2\left[\theta_{10}(x) + \frac{\theta_{11}(x)}{\lambda}\right] + 2\sqrt{a}\lambda\left[\theta'_{10}(x) + \frac{\theta'_{11}(x)}{\lambda}\right] + \left[\theta''_{10}(x) + \frac{\theta''_{11}(x)}{\lambda}\right]$$

$$-a\left[\theta_{10}(x) + \frac{\theta_{11}(x)}{\lambda}\right]\left[\lambda^2 + b\lambda + c + a^{-1}h(\lambda)q^{-1}(\lambda)\right]$$

\[ = \lambda [2 \sqrt{a} \theta'_{10}(x) - ab \theta_{10}(x)] + [2 \sqrt{a} \theta'_{11}(x) - ab \theta_{11}(x) + \theta''_{10}(x) - ac \theta_{10}(x)] + \lambda^{-1} F_1(x, \lambda) \]

and

\[
e^{\sqrt{a} \lambda x} D(\tilde{\theta}_2(x, \lambda)) = a \lambda^2 \left[ \theta_{20}(x) + \frac{\theta_{21}(x)}{\lambda} \right] - 2 \sqrt{a} \lambda \left[ \theta'_{20}(x) + \frac{\theta'_{21}(x)}{\lambda} \right] + \left[ \theta''_{20}(x) + \frac{\theta''_{21}(x)}{\lambda} \right] - a \left[ \theta_{20}(x) + \frac{\theta_{21}(x)}{\lambda} \right] \left( \lambda^2 + b \lambda + c + a^{-1} h(\lambda) q^{-1}(\lambda) \right) = -\lambda [2 \sqrt{a} \theta'_{20}(x) + ab \theta_{20}(x)] - [2 \sqrt{a} \theta'_{21}(x) + ab \theta_{21}(x) - \theta''_{20}(x) + ac \theta_{20}(x)] + \lambda^{-1} F_2(x, \lambda) \]

where

\[ F_i(x, \lambda) = \theta''_{i1}(x) - ac \theta_{i1}(x) - [\lambda \theta_{i0}(x) + \theta_{i1}(x)] h(\lambda) q^{-1}(\lambda), \quad i = 1, 2 \]

in which

\[ |[\lambda \theta_{i0}(x) + \theta_{i1}(x)] h(\lambda) q^{-1}(\lambda)| \leq M, \quad |F_i(x, \lambda)| \leq M \quad \forall x \in [0, 1] \]

for some positive constant \( M \). Thus, letting the coefficients of \( \lambda^1 \) and \( \lambda^0 \) be zero give

\[ 2 \sqrt{a} \theta'_{10}(x) - ab \theta_{10}(x) = 0, \quad 2 \sqrt{a} \theta'_{20}(x) + ab \theta_{20}(x) = 0 \]

and

\[ 2 \sqrt{a} \theta'_{11}(x) - ab \theta_{11}(x) + \theta''_{10}(x) - ac \theta_{10}(x) = 0 \]
\[ 2 \sqrt{a} \theta'_{21}(x) + ab \theta_{21}(x) - \theta''_{20}(x) + ac \theta_{20}(x) = 0 \]

Now, use the conditions \( \theta_{i0}(0) = 1, \theta_{i1}(0) = 0, i = 1, 2 \), to obtain

\[ \theta_{10}(x) = e^{(1/2) \sqrt{ab} x}, \quad \theta_{11}(x) = -\frac{1}{8} \sqrt{a} (b^2 - 4c) x e^{(1/2) \sqrt{ab} x} \]

and

\[ \theta_{20}(x) = e^{(-1/2) \sqrt{ab} x}, \quad \theta_{21}(x) = \frac{1}{8} \sqrt{a} (b^2 - 4c) x e^{(-1/2) \sqrt{ab} x} \]

These are (37). When \( |\lambda| \) is large enough, we obtain the linearly independent asymptotic fundamental solutions of \( \theta''(x) - a \theta(x)(\lambda^2 + b \lambda + c + a^{-1} h(\lambda) q^{-1}(\lambda)) = 0 \) given by (36) (see [9]):

\[ \theta_1(x) = e^{\sqrt{a} \lambda x} \left[ \theta_{10}(x) + \frac{\theta_{11}(x)}{\lambda} \right] + O(\lambda^{-2}) \]

and

\[ \theta_2(x) = e^{-\sqrt{a} \lambda x} \left[ \theta_{20}(x) + \frac{\theta_{21}(x)}{\lambda} \right] + O(\lambda^{-2}) \]

The proof is complete. \( \square \)
Let $\lambda \neq 0$ and $\lambda \notin \Lambda$, and let

$$\theta(x) = e^{\theta_1(x)} + f \theta_2(x)$$

where $\theta_i(x), i = 1, 2$ are given by (36). Then, substitute the above into the boundary conditions of (33) to obtain

$$\Delta(\lambda)[e, f] = 0$$

where

$$\Delta(\lambda) = \begin{bmatrix} 1 & 1 \\ \theta_1(1) & \theta_2(1) \end{bmatrix}$$

Hence, (33) has non-trivial solution if and only if

$$\det(\Delta(\lambda)) = 0$$

and the eigenvalues of (33) are the zeros of (42). Notice that

$$\det(\Delta(\lambda)) = \theta_2(1) - \theta_1(1)$$

$$= e^{-\sqrt{a} \lambda} \{ \theta_{20}(1) + \theta_{21}(1) \lambda^{-1} \} - e^{\sqrt{a} \lambda} \{ \theta_{10}(1) + \theta_{11}(1) \lambda^{-1} \} + C(\lambda^{-2})$$

where

$$d := \frac{1}{8} \sqrt{a} (b^2 - 4c)$$

Hence, $\det(\Delta(\lambda)) = 0$ produces

$$e^{-\sqrt{a} \lambda} \{ e^{(-1/2) \sqrt{a} b} + d e^{(-1/2) \sqrt{a} b} \lambda^{-1} \} - e^{\sqrt{a} \lambda} \{ e^{(1/2) \sqrt{a} b} - d e^{(1/2) \sqrt{a} b} \lambda^{-1} \} + C(\lambda^{-2}) = 0$$

which further leads to

$$e^{-\sqrt{a} \lambda} e^{(-1/2) \sqrt{a} b} - e^{\sqrt{a} \lambda} e^{(1/2) \sqrt{a} b} + C(\lambda^{-1}) = 0$$

Finally, since the solutions of $e^{2 \sqrt{a} \lambda + \sqrt{a} b} - 1 = 0$ are

$$\tilde{\lambda}_n = -\frac{1}{2} b + \frac{\sqrt{a}}{a} n \pi i, \quad n \in \mathbb{Z}$$

Apply Rouché’s theorem to (44) to give the solutions of (44):

$$\tilde{\lambda}_n = -\frac{1}{2} b + \frac{\sqrt{a}}{a} n \pi i + C(n^{-1}), \quad n \in \mathbb{Z}$$

**Theorem 3.1**

The eigenvalues of (33) have the following asymptotic expressions:

$$\lambda_n = -\frac{1}{2} b + \frac{\sqrt{a}}{a} n \pi i + C(n^{-1}), \quad n \in \mathbb{Z}$$
in particular,
\[ \text{Re}\, \lambda_n \to -\frac{1}{2}b = -\frac{1}{2} \sum_{i=1}^{N} a_i^2 b_i < 0 \text{ as } |n| \to \infty \] (46)

that is, \( \text{Re}\, \lambda = -\frac{1}{2}b \) is the asymptote of the eigenvalues \( \lambda_n \) given by (45). Here, \( b \) is given by (28). Furthermore, the corresponding eigenfunctions \( \theta_n(x), n \in \mathbb{Z} \) have the asymptotic expressions
\[ \theta_n(x) = \sin n\pi x + \mathcal{O}(n^{-1}) \] (47)
and
\[ \lambda_n^{-1} \theta_n'(x) = -i\sqrt{a} \cos n\pi x + \mathcal{O}(n^{-1}) \] (48)

Moreover, \( \{\theta_n(x), n \in \mathbb{Z}\} \) and \( \{\lambda_n^{-1} \theta_n'(x), n \in \mathbb{Z}\} \) are approximately normalized in \( L^2(0,1) \) in the sense that there exist positive constants \( c_1 \) and \( c_2 \) independent of \( n \) such that for \( n \in \mathbb{Z} \),
\[ c_1 \leq \|\theta_n\|_{L^2}, \quad \|\lambda_n^{-1} \theta_n'\|_{L^2} \leq c_2 \] (49)

**Proof**
Equation (45) has been proved. Only proofs for (47)–(49) are needed. Since \( \lambda \neq 0, \lambda \notin \Lambda \), in view of (41), (45), Lemma 3.4 and some facts in linear algebra, the eigenfunction \( \theta \) corresponding to the eigenvalue \( \lambda \) is given by
\[ \theta(\lambda, x) = \det \begin{bmatrix} 1 & 1 \\ \theta_1(x) & \theta_2(x) \end{bmatrix} = \theta_2(x) - \theta_1(x) = e^{(-1/2)\sqrt{ab}x} e^{-\sqrt{\alpha} \lambda x} - e^{(1/2)\sqrt{ab}x} e^{\sqrt{\alpha} \lambda x} + \mathcal{O}(\lambda^{-1}) \]
\[ \theta'(\lambda, x) = \det \begin{bmatrix} 1 & 1 \\ \theta_1'(x) & \theta_2'(x) \end{bmatrix} = \left( -\frac{1}{2} \sqrt{ab} - \sqrt{\alpha} \lambda \right) e^{(-1/2)\sqrt{ab}x} e^{-\sqrt{\alpha} \lambda x} - \left( -\frac{1}{2} \sqrt{ab} + \sqrt{\alpha} \lambda \right) e^{(1/2)\sqrt{ab}x} e^{\sqrt{\alpha} \lambda x} + \mathcal{O}(\lambda^{-1}) \]

Owing to the fact of (45) that
\[ \frac{1}{2} \sqrt{ab} + \sqrt{\alpha} \lambda_n = n\pi i + \mathcal{O}(n^{-1}), \quad n \in \mathbb{Z} \]

Equations (47) and (48) are thus proved by taking
\[ \theta_n(x) = \frac{1}{2}i\theta(\lambda_n, x) \]

Finally,
\[ \|\theta_n\|_{L^2} = \int_0^1 \sin^2 n\pi x \, dx + \mathcal{O}(n^{-1}) = \frac{1}{2} + \mathcal{O}(n^{-1}) \] (50)
and
\[
\|\lambda_n^{-1} \partial_n'\|_{L^2} = \int_0^1 (-i \sqrt{a} \cos n \pi x) (-i \sqrt{a} \cos n \pi x) \, dx + \mathcal{O}(n^{-1}) = \frac{a}{2} + \mathcal{O}(n^{-1}) \tag{51}
\]
These give (49). The proof is complete. 

Remark 3.1
From the asymptotic expression (45) and (46), the asymptote
\[
-\frac{1}{2} b = -\frac{1}{2} \frac{\sum_{i=1}^N a_i^2 b_i}{\sum_{i=1}^N a_i^2} < 0
\]
is half the average of the exponents \(-b_i, i = 1, 2, \ldots, N\), with the weights \(a_i^2 / (\sum_{i=1}^N a_i^2)\).

The following result is the consequence of Lemmas 3.1, 3.2 and Theorem 3.1.

Corollary 3.1
Let \(\mathcal{A}\) be defined by (11). Then \(\mathcal{A}\) has the eigenvalues
\[
\{-b_i, i = 1, 2, \ldots, N\} \cup \{\lambda_n, n \in \mathbb{Z}\} \tag{52}
\]
where \(\lambda_n\) has the asymptotic expression (45). All eigenvalues with large modulus are algebraically simple. Moreover, \(\mathcal{A}\) has the eigenfunction \(e_{i+1}, i = 1, 2, \ldots, N\) corresponding to \(-b_i\), and the eigenfunctions corresponding to \(\lambda_n, n \in \mathbb{Z}\) are given by
\[
[\theta_n, (w_1)_n, \ldots, (w_N)_n] \tag{53}
\]
where \(\theta_n\) is given by (47) and
\[
(w_i)_n = \frac{a_i \partial_n'(x)}{\lambda_n + b_i} = -a_i \sqrt{a} \cos n \pi x + \mathcal{O}(n^{-1}) \tag{54}
\]
Proof
We only need to prove the algebraic simplicity for the eigenvalues since others are claimed by Lemma 3.1 and Theorem 3.1. From the expression (78) given later in Section 4, the order of each \(\lambda \in \sigma_p(\mathcal{A}) \setminus \{-b_i, i = 1, 2, \ldots, N\}\), as a pole of \(R(\lambda, \mathcal{A})\) with sufficiently large modulus is less than or equal to the multiplicity of \(\lambda\) as a zero of the entire function \(\sinh(\sqrt{a} \mu(\lambda))\), where \(\mu(\lambda)\) is given by (68) later on. Since it is easy to see that \(\lambda\) is geometrically simple and from (44) all zeros of \(\sinh(\sqrt{a} \mu(\lambda)) = 0\), which has the asymptotic expression (44) with large moduli are simple, the result then follows from the formula: \(m_a \leq p \cdot m_g\) (see e.g. [11, p.148]), where \(p\) denotes the order of the pole of the resolvent operator, and \(m_a, m_g\) denote the algebraic and geometric multiplicities, respectively.

In order to investigate the residual and continuous spectrum of \(\mathcal{A}\), we need the adjoint operator \(\mathcal{A}^*\). Since \(\mathcal{A}\) involves only the first-order differential operator, we need a little bit careful analysis
to get $\mathcal{A}^\ast$. Notice that $\mathcal{A}$ is a bounded perturbation of the following operator:

$$\mathcal{A}_0 \begin{bmatrix} \theta \\
w_1 \\
\vdots \\
w_N \end{bmatrix}^T = \begin{bmatrix} \frac{d}{dx} \sum_{i=1}^{N} a_i w_i \\
1_{\mathcal{A}_0}' \\
\vdots \\
N_{\mathcal{A}_0}' \end{bmatrix}^T \forall \begin{bmatrix} \theta \\
w_1 \\
\vdots \\
w_N \end{bmatrix}^T \in D(\mathcal{A}_0) = D(\mathcal{A}) \quad (55)$$

**Lemma 3.5**

Let $\mathcal{A}_0$ be defined by (55). Then $\mathcal{A}_0$ is a skew-adjoint operator in $\mathcal{H}$ and $(\mathcal{A}_0 - i\xi I)^{-1}$ exists and is bounded on $\mathcal{H}$ for $\xi = \sqrt{\alpha/\alpha}$.

**Proof**

We first show that $(\mathcal{A}_0 - i\xi I)^{-1}$ exists and is bounded on $\mathcal{H}$.

Solve $(\mathcal{A}_0 - i\xi I)F = G = [g, f_1, \ldots, f_N]$ for $F = [\theta, w_1, \ldots, w_N] \in D(\mathcal{A}_0)$, that is,

$$\begin{cases}
d \sum_{j=1}^{N} a_j w_j(x) - i\xi \theta(x) = g(x) \\
a_j \theta'(x) - i\xi w_j(x) = f_j(x), \quad j = 1, 2, \ldots, N \\
\theta(0) = \theta(1) = 0
\end{cases} \quad (56)$$

to get

$$w_j(x) = \frac{1}{i\xi} [a_j \theta'(x) - f_j(x)], \quad \sum_{j=1}^{N} a_j w_j(x) = \frac{1}{i\xi} \left[ \sum_{j=1}^{N} a_j f_j(x) \right] \quad (57)$$

where $a = (\sum_{i=1}^{N} a_i^2)^{-1}$ is defined by (28). Let

$$w(x) = \sum_{j=1}^{N} a_j w_j(x), \quad f(x) = \sum_{j=1}^{N} a_j f_j(x)$$

Then (56) and (57) show that

$$\begin{cases}
w'(x) = i\xi \theta(x) + g(x) \\
\theta'(x) = i\alpha \omega(x) + af(x) \\
\theta(0) = \theta(1) = 0
\end{cases} \quad (58)$$

The first two equations can be rewritten as the following system of the first-order differential equations:

$$\frac{d}{dx} \begin{bmatrix} w \\
\theta \end{bmatrix} = i\xi \begin{bmatrix} 0 & 1 \\
a & 0 \end{bmatrix} \begin{bmatrix} w \\
\theta \end{bmatrix} + \begin{bmatrix} g \\
af \end{bmatrix} \quad (58)$$
Let

\[ A = \begin{bmatrix} 0 & 1 \\ a & 0 \end{bmatrix} \]  \hspace{1cm} (59) \]

Then, it has

\[ e^{Ax} = \begin{bmatrix} \cosh(\sqrt{a}x) & (\sqrt{a})^{-1}\sinh(\sqrt{a}x) \\ \sqrt{a}\sinh(\sqrt{a}x) & \cosh(\sqrt{a}x) \end{bmatrix} \]

The solution of (58) is then found to be

\[
\begin{bmatrix} w(x) \\ \theta(x) \end{bmatrix} = e^{i\xi A} \begin{bmatrix} w(0) \\ \theta(0) \end{bmatrix} + \int_0^x e^{i\xi A(x-\tau)} \begin{bmatrix} g(\tau) \\ af(\tau) \end{bmatrix} d\tau
\]

\[
= e^{i\xi A} \begin{bmatrix} w(0) \\ 0 \end{bmatrix} + \int_0^x e^{i\xi A(x-\tau)} \begin{bmatrix} g(\tau) \\ af(\tau) \end{bmatrix} d\tau
\]

\[
= \begin{bmatrix} w(0)\cos(\xi \sqrt{a}x) \\ i\sqrt{a}w(0)\sin(\xi \sqrt{a}x) \end{bmatrix}
\]

\[
+ \int_0^x \begin{bmatrix} \cos(\xi \sqrt{a}(x-\tau)) & i(\sqrt{a})^{-1}\sin(\xi \sqrt{a}(x-\tau)) \\ i\sqrt{a}\sin(\xi \sqrt{a}(x-\tau)) & \cos(\xi \sqrt{a}(x-\tau)) \end{bmatrix} \begin{bmatrix} g(\tau) \\ af(\tau) \end{bmatrix} d\tau
\]

\[
= \begin{bmatrix} w(0)\cos(\xi \sqrt{a}x) \\ i\sqrt{a}w(0)\sin(\xi \sqrt{a}x) \end{bmatrix}
\]

\[
+ \int_0^x \begin{bmatrix} \cos(\xi \sqrt{a}(x-\tau))g(\tau) + i\sqrt{a}\sin(\xi \sqrt{a}(x-\tau))f(\tau) \\ i\sqrt{a}\sin(\xi \sqrt{a}(x-\tau))g(\tau) + a\cos(\xi \sqrt{a}(x-\tau))f(\tau) \end{bmatrix} d\tau
\]

that is,

\[
w(x) = w(0)\cos(\xi \sqrt{a}x) + \int_0^x [\cos(\xi \sqrt{a}(x-\tau))g(\tau) + i\sqrt{a}\sin(\xi \sqrt{a}(x-\tau))f(\tau)] d\tau
\]

and

\[
\theta(x) = i\sqrt{a}w(0)\sin(\xi \sqrt{a}x) + \int_0^x [i\sqrt{a}\sin(\xi \sqrt{a}(x-\tau))g(\tau) + a\cos(\xi \sqrt{a}(x-\tau))f(\tau)] d\tau
\]
where
\[
 w(0) = -\frac{1}{\sin(\xi \sqrt{\alpha})} \int_0^1 [\sin(\xi \sqrt{\alpha}(1-\tau))g(\tau) - i \sqrt{\alpha} \cos(\xi \sqrt{\alpha}(1-\tau))f(\tau)] d\tau
\]

Since \( \sin(\xi \sqrt{\alpha}) \neq 0 \), we see that \( \theta(x) \) is uniquely determined. Once \( \theta(x) \) is known, the \( w_i(x) \) is also uniquely determined by (57)
\[
 w_j(x) = \frac{1}{i\xi} [a_j\theta'(x) - f_j(x)] = \frac{1}{i\xi} [ia_j\theta w(x) + a_j f(x) - f_j(x)], \quad j = 1, 2, \ldots, N
\]

Hence, \( (\mathcal{A}_0 - i\xi I)^{-1} \) exists and is bounded on \( \mathcal{H} \).

Now, a direct computation shows that \( \xi I + \mathcal{A}_0 \) is symmetric in \( \mathcal{H} \) and \( (\xi I + i\mathcal{A}_0)^{-1} = -i(\mathcal{A}_0 - i\xi I)^{-1} \), it follows from Theorem 13.11 of [12] that \( \xi I + i\mathcal{A}_0 \) is self-adjoint. Thus, \( \mathcal{A}_0 \) is skew adjoint. The proof is complete.

Since \( \mathcal{A} \) is a bounded perturbation of \( \mathcal{A}_0 \) and \( \mathcal{A}_0 \) is skew adjoint, we have immediately that

\[
 A^* = \begin{bmatrix} \theta \\ w_1 \\ \vdots \\ w_N \end{bmatrix}^T = \begin{bmatrix} \frac{d}{dx} \sum_{i=1}^N a_i w_i \\ a_1 \theta' + b_1 w_1 \\ \vdots \\ a_N \theta' + b_N w_N \end{bmatrix}^T, \quad D(A^*) = \begin{bmatrix} \theta \\ w_1 \\ \vdots \\ w_N \end{bmatrix}^T \in H_0^1(0, 1), \quad \sum_{i=1}^N a_i w_i \in H^1(0, 1)
\]

(60)

**Theorem 3.2**

Let \( \mathcal{A} \) be defined by (11). Then \( \sigma_\tau(\mathcal{A}) = 0 \), where \( \sigma_\tau(\mathcal{A}) \) denotes the set of residual spectrum of \( \mathcal{A} \).

**Proof**

Since \( \lambda \in \sigma_r(\mathcal{A}) \), \( \bar{\lambda} \in \sigma_p(\mathcal{A}^*) \), the proof will be accomplished if we can show that \( \sigma_p(\mathcal{A}) = \sigma_p(\mathcal{A}^*) \).

This is because obviously the eigenvalues of \( \mathcal{A} \) are symmetric on the real axis. From (60), the eigenvalue problem \( \mathcal{A}^* F = \lambda F \), where \( F = [\theta, w_1, \ldots, w_N] \in D(A^*) \) reads:

\[
\begin{align*}
\dot{\lambda} \theta(x) + \frac{d}{dx} \sum_{i=1}^N a_i w_i(x) &= 0, & 0 < x < 1 \\
\dot{\lambda} w_i(x) + a_i \theta'(x) + b_i w_i(x) &= 0, & i = 1, 2, \ldots, N, \quad 0 < x < 1 \\
\theta(0) &= \theta(1) = 0
\end{align*}
\]

(61)

Equation (61) is the same as (19) by setting \( \tilde{\theta} = -\theta \). Hence, \( \mathcal{A}^* \) has the same eigenvalues with \( \mathcal{A} \). The proof is complete.
Theorem 3.3
Let \( \Lambda \) be defined by (24). Then
\[
\sigma_c(\mathcal{A}) = \Lambda \quad \text{where } \sigma_c(\mathcal{A}) \text{ is the set of the continuous spectrum of } \mathcal{A}
\] (62)

Proof
Let \( \lambda \notin \sigma_p(\mathcal{A}) \). For any \( G = [g, f_1, \ldots, f_N] \in \mathcal{H} \), since \( -b_i \in \sigma_p(\mathcal{A}), i = 1, 2, \ldots, N \), solve \( (\lambda I - \mathcal{A})[\theta, w_1, \ldots, w_N] = G \); that is,
\[
\begin{cases}
\lambda \theta(x) - \frac{d}{dx} \sum_{i=1}^{N} a_i w_i(x) = g(x) \\
\lambda w_i(x) - a_i \theta'(x) + b_i w_i(x) = f_i(x), \quad i = 1, 2, \ldots, N \\
\theta(0) = \theta(1) = 0
\end{cases}
\] (63)
to get
\[
w_i(x) = \frac{a_i}{\lambda + b_i} \theta'(x) + \frac{1}{\lambda + b_i} f_i(x), \quad \sum_{i=1}^{N} a_i w_i(x) = \theta'(x) \sum_{i=1}^{N} \frac{a_i^2}{\lambda + b_i} + \sum_{i=1}^{N} \frac{a_i}{\lambda + b_i} f_i(x)
\] (64)
Let
\[
w(x) = \sum_{i=1}^{N} a_i w_i(x), \quad f(\lambda, x) = \sum_{i=1}^{N} \frac{a_i}{\lambda + b_i} f_i(x)
\] (65)
We claim that \( \Lambda \subseteq \sigma_c(\mathcal{A}) \). In fact, when \( \lambda \in \Lambda \), it has
\[
\sum_{i=1}^{N} \frac{a_i^2}{\lambda + b_i} = 0
\]
By (64) and (65), it has
\[
w(x) = f(\lambda, x)
\]
Since \( w(x) \in H^1(0, 1) \), the above identity holds true unless \( f(\lambda, x) \in H^1(0, 1) \). This shows that \( \lambda \notin \rho(\mathcal{A}) \), or
\[
\Lambda \subseteq \sigma_c(\mathcal{A})
\] (66)
by Lemma 3.2 and Theorem 3.2.

Now we show that \( \sigma_c(\mathcal{A}) \subseteq \Lambda \), or equivalently, any \( \lambda \notin \sigma_p(\mathcal{A}) \cup \Lambda \) belongs to \( \rho(\mathcal{A}) \). To do this, suppose that \( \lambda \notin \sigma_p(\mathcal{A}) \cup \Lambda \). We write (63) as
\[
\begin{cases}
w'(x) = \lambda \theta(x) - g(x) \\
\theta'(x) = \mu(\lambda) w(x) - \mu(\lambda) f(\lambda, x) \\
\theta(0) = \theta(1) = 0
\end{cases}
\] (67)
where

$$
\mu(\lambda) = \left( \sum_{i=1}^{N} \frac{a_i^2}{\lambda + b_i} \right)^{-1}
$$

for any $\lambda \not\in \sigma_p(\mathcal{A}) \cup \Lambda$  \hspace{1cm} (68)

Hence, the first two equations (67) can be rewritten as the following system of the first-order differential equations:

$$
\frac{d}{dx} \begin{bmatrix} w \\ \theta \end{bmatrix} = \begin{bmatrix} 0 & \lambda \\ \mu(\lambda) & 0 \end{bmatrix} \begin{bmatrix} w \\ \theta \end{bmatrix} - \begin{bmatrix} g \\ \mu(\lambda) f \end{bmatrix}
$$

(69)

Let

$$
A(\lambda) = \begin{bmatrix} 0 & \lambda \\ \mu(\lambda) & 0 \end{bmatrix}
$$

(70)

Then it has

$$
e^{A(\lambda)x} = \begin{bmatrix} a_{11}(\lambda, x) & a_{12}(\lambda, x) \\ a_{21}(\lambda, x) & a_{22}(\lambda, x) \end{bmatrix}
$$

(71)

where

$$
\begin{align*}
a_{11}(\lambda, x) & = \cosh(\sqrt{\lambda} \mu(\lambda)x), & a_{12}(\lambda, x) & = \frac{\sqrt{\lambda}}{\sqrt{\mu(\lambda)}} \sinh(\sqrt{\lambda} \mu(\lambda)x) \\
a_{21}(\lambda, x) & = \frac{\sqrt{\mu(\lambda)}}{\sqrt{\lambda}} \sinh(\sqrt{\lambda} \mu(\lambda)x), & a_{22}(\lambda, x) & = \cosh(\sqrt{\lambda} \mu(\lambda)x)
\end{align*}
$$

(72)

The solution of (69) is then found to be

$$
\begin{bmatrix} w(x) \\ \theta(x) \end{bmatrix} = e^{A(\lambda)x} \begin{bmatrix} w(0) \\ \theta(0) \end{bmatrix} - \int_{0}^{x} e^{A(\lambda)(x-\tau)} \begin{bmatrix} g(\tau) \\ \mu(\lambda) f(\lambda, \tau) \end{bmatrix} d\tau
$$

$$
= \begin{bmatrix} w(0)a_{11}(\lambda, x) \\ w(0)a_{21}(\lambda, x) \end{bmatrix} - \int_{0}^{x} \begin{bmatrix} a_{11}(\lambda, x-\tau)g(\tau) + \mu(\lambda)a_{12}(\lambda, x-\tau)f(\lambda, \tau) \\ a_{21}(\lambda, x-\tau)g(\tau) + \mu(\lambda)a_{22}(\lambda, x-\tau)f(\lambda, \tau) \end{bmatrix} d\tau
$$

that is,

$$
\begin{bmatrix} w(x) = w(0)a_{11}(\lambda, x) - \int_{0}^{x} [a_{11}(\lambda, x-\tau)g(\tau) + \mu(\lambda)a_{12}(\lambda, x-\tau)f(\lambda, \tau)] d\tau \end{bmatrix}
$$

(73)
and
\[ \theta(x) = w(0)a_{21}(\lambda, x) - \int_0^x [a_{21}(\lambda, x - \tau)g(\tau) + \mu(\lambda)a_{22}(\lambda, x - \tau)f(\lambda, \tau)]d\tau \]  \quad (74)

When \( G = [g, f_1, \ldots, f_N] = 0 \), Equations (73) and (74) reduce to the eigenvalue problem
\[ w(x) = w(0)a_{11}(\lambda, x), \quad \theta(x) = w(0)a_{21}(\lambda, x) \]

Hence, when \( \lambda \in \sigma_p(A), \lambda \neq -b_i, i = 1, 2, \ldots, N \) if and only if \( a_{21}(\lambda, 1) = 0 \), that is,
\[ a_{21}(\lambda, 1) = \frac{\sqrt{\mu(\lambda)}}{\sqrt{\lambda}} \sinh(\sqrt{\lambda \mu(\lambda)}) = 0 \]  \quad (75)

which yields
\[ \sinh(\sqrt{\lambda \mu(\lambda)}) = 0 \]  \quad (76)

This is the characteristic determinant of \( \mathcal{A} \), which has the asymptotic form given by (44).

Now, since \( \lambda \notin \sigma_p(A) \cup \Lambda \), we have
\[ w(0) = \frac{1}{a_{21}(\lambda, 1)} \int_0^1 [a_{21}(\lambda, 1 - \tau)g(\tau) + \mu(\lambda)a_{22}(\lambda, 1 - \tau)f(\lambda, \tau)]d\tau \]

Hence, \( \theta(x) \) is uniquely determined by (74) and \( \theta' \in L^2(0, 1) \). Once \( \theta'(x) \) is known, the \( w_i(x) \) is also uniquely determined by (64):
\[ w_i(x) = \frac{a_i}{\lambda + b_i} \theta'(x) + \frac{1}{\lambda + b_i} f_i(x), \quad i = 1, 2, \ldots, N \]

Hence, \((\lambda I - \mathcal{A})^{-1}\) exists and is bounded. Therefore, \( \lambda \in \rho(\mathcal{A}) \). The proof is complete. \( \square \)

4. SPECTRUM-DETERMINED GROWTH CONDITION AND EXPONENTIAL STABILITY

Now we are in a position to consider the main result of this paper, the so-called spectrum-determined growth condition for system (12), which is one of the most difficult problems for infinite-dimensional systems. Our proof is based on the following characterization condition (see [11, Corollary 3.40]).

**Lemma 4.1**

Let \( T(t) \) be a \( C_0 \)-semigroup on a Hilbert space \( H \) with its generator \( A \). Let \( \omega(A) \) be the growth bound of \( T(t) \) and
\[ s(A) := \sup \{ \text{Re} \lambda | \lambda \in \sigma(A) \} \]
be the spectral bound of $\mathbf{A}$. Then

$$\omega(\mathbf{A}) = \inf \left\{ \omega > s(\mathbf{A}) \left| \sup_{\tau \in \mathbb{R}} \| R(\sigma + i\tau, \mathbf{A}) \| < M_\sigma < \infty, \ \forall \sigma \geq \omega \right. \right\}$$

We also need the Lemma 1.2 of [13] (see also [14]).

**Lemma 4.2**

Let

$$D(\lambda) := 1 + \sum_{i=1}^{n} Q_i(\lambda)e^{zi}$$

where $Q_i$ are polynomials of $\lambda$, $z_i$ are some complex numbers and $n$ is a positive integer. Then for all $\lambda$ outside those circles of radius $\varepsilon > 0$ that centered at the zeros of $D(\cdot)$, one has

$$|D(\lambda)| \geq C(\varepsilon) > 0$$

for some constant $C(\varepsilon)$ that depends only on $\varepsilon$.

**Lemma 4.3**

Let $\lambda$ be defined by (24) and $\lambda \not\in \Lambda$. Let

$$D_0(\lambda) := \sinh(\sqrt{\mu(\lambda)})$$

where $\mu(\lambda)$ is given by (68). Then as indicated by (76), all eigenvalues $\lambda_n$ claimed by (45) are zeros of $D_0(\lambda)$. Moreover, for all those $\lambda \not\in \Lambda$ that are outside those circles centered at $\lambda_n$ with radius $\varepsilon > 0$, it has

$$|D_0(\lambda)| \geq e^{\sqrt{a}Re(\lambda)} C_0(\varepsilon) > 0$$

for some constant $C_0(\varepsilon)$ that depends only on $\varepsilon$, where $a$ is given by (28).

**Proof**

By (44), it has

$$D_0(\lambda) = -e^{\sqrt{a} \lambda} e^{(1/2)\sqrt{ab}} \left[ 1 - e^{-2\sqrt{a} \lambda} e^{-\sqrt{ab}} + O(\lambda^{-1}) \right]$$

The result then follows from Lemma 4.2. $\square$

**Theorem 4.1**

Let $\mathcal{A}$ be defined by (11). Then the spectrum-determined growth condition holds true for $e^{\mathcal{A}t}$:

$$s(\mathcal{A}) = \omega(\mathcal{A}).$$

**Proof**

By Lemma 4.1, the proof will be accomplished if we can show that for any $\lambda \neq 0$ and $\lambda = \sigma + i\tau \in \mathbb{C}$ with $\sigma \geq \omega > s(\mathcal{A})$ and $\tau \in \mathbb{R}$, there is a constant $M_\sigma$ such that

$$\sup_{\tau \in \mathbb{R}} \| R(\sigma + i\tau, \mathcal{A}) \| \leq M_\sigma < \infty$$

(77)
Let $\lambda = \sigma + \tau \in \mathbb{C}$ with $\sigma \geq \omega > s(\mathcal{A})$ and $\tau \in \mathbb{R}$. For any $G = [g, f_1, \ldots, f_N] \in \mathcal{H}$, from the proof of the second part of Theorem 3.3, $F = R(\lambda, \mathcal{A})G = [\theta, w_1, \ldots, w_N] \in D(\mathcal{A})$ satisfies

\[
\begin{align*}
\theta(x) &= w(0)a_{21}(\lambda, x) - \int_0^x [a_{21}(\lambda, x - \tau)g(\tau) + \mu(\lambda)a_{22}(\lambda, x - \tau)f(\lambda, \tau)]d\tau \\
w_j(x) &= \frac{a_j}{\lambda + b_j} \theta'(x) + \frac{1}{\lambda + b_j} f_j(x), \quad j = 1, 2, \ldots, N \\
f(\lambda, x) &= \sum_{j=1}^N \frac{a_j}{\lambda + b_j} f_j(x), \quad \mu(\lambda) = \left( \sum_{j=1}^N \frac{a_j^2}{\lambda + b_j} \right)^{-1} \\
\theta'(x) &= \mu(\lambda) w(x) - \mu(\lambda) f(\lambda, x) \\
w(x) &= w(0)a_{11}(\lambda, x) - \int_0^x [a_{11}(\lambda, x - \tau)g(\tau) + \mu(\lambda)a_{12}(\lambda, x - \tau)f(\lambda, \tau)]d\tau \\
w(0) &= \frac{1}{a_{21}(\lambda, 1)} \int_0^1 [a_{21}(\lambda, 1 - \tau)g(\tau) + \mu(\lambda)a_{22}(\lambda, 1 - \tau)f(\lambda, \tau)]d\tau
\end{align*}
\]

(78)

Here, $a_{jk}(\lambda, x), 1 \leq j, k \leq 2$ are given by (72). First, by Lemma 3.3, we obtain

\[\mu(\lambda) = a\lambda + ab + \mathcal{O}(\lambda^{-1})\]

where $a, b$ are positive constants given by (28). Thus, it is easy to see that there is a positive constant $M_{1\sigma}$ such that

\[\sup_{\tau \in \mathbb{R}} \|\mu(\lambda) f(\lambda, \cdot)\|_{L^2} \leq M_{1\sigma} \sum_{j=1}^N \|f_j\|_{L^2} < \infty \]

(79)

Second, since

\[
\begin{align*}
\int_0^1 e^{\pm a\lambda x} e^{\pm a\bar{\lambda} x} dx &= \int_0^1 e^{\pm a(\lambda + \bar{\lambda}) x} dx = \int_0^1 e^{\pm 2a\sigma x} dx = \pm \frac{1}{2a\sigma} (e^{\pm 2a\sigma} - 1) \\
\int_0^1 e^{\pm a\lambda x} e^{\mp a\bar{\lambda} x} dx &= \int_0^1 e^{\pm a(\lambda - \bar{\lambda}) x} dx = \int_0^1 e^{\pm 2ia\tau x} dx = \begin{cases} 1, & \tau = 0 \\ \pm \frac{1}{2ia\tau} (e^{\pm 2ia\tau} - 1), & \tau \neq 0 \end{cases}
\end{align*}
\]

there is a positive constant $M_{2\sigma}$ such that

\[\sup_{\tau \in \mathbb{R}} \|a_{jk}(\lambda, \cdot)\|_{L^2} \leq M_{2\sigma} < \infty, \quad 1 \leq j, k \leq 2\]

(80)

Third, by Lemma 3.1, Proposition 3.1, Theorems 3.2 and 3.3, it has

\[s(\mathcal{A}) = \sup \{\text{Re } \lambda | \lambda \in \sigma(\mathcal{A})\} = \sup \{\text{Re } \lambda | \lambda \in \sigma_p(\mathcal{A})\}\]

(81)

Define

\[\varepsilon_\sigma = \inf_{\lambda_n \in \sigma_p(\mathcal{A}), \tau \in \mathbb{R}} |\lambda_n - \sigma - i\tau|\]
By Lemma 4.3, there is a positive constant $C_0(\varepsilon_\sigma)$ depending on $\sigma$ only such that

$$|D_0(\lambda)| = |\sinh(\sqrt{\lambda + \mu(\lambda)})| \geq \sqrt{\lambda + \mu(\lambda)} C_0(\varepsilon_\sigma) > 0$$

Hence, by (75), there is a positive constant $M_{3\sigma}$ depending on $\sigma$ only such that

$$\left| \frac{1}{a_{21}(\lambda)} \right| \leq M_{3\sigma} < \infty$$

Hence, by the Cauchy–Schwartz inequality, it follows from (79), (80) and the last equation of (78) that there is a positive constant $M_{4\sigma}$ such that

$$\sup_{\tau \in \mathbb{R}} |w(0)| \leq M_{4\sigma} \|G\|_{\mathcal{H}} < \infty$$

Finally, from (78), there is a positive constant $M_\sigma$ such that

$$\sup_{\tau \in \mathbb{R}} \|F\|_{\mathcal{H}} = \sup_{\tau \in \mathbb{R}} \left\{ \|\theta\|_{L^2} + \sum_{j=1}^{N} \|f_j\|_{L^2} \right\} \leq M_{\sigma} \|G\|_{\mathcal{H}} < \infty$$

This gives (77). The proof is complete.

The following result gives a strongly exponential stability for the system (1).

**Theorem 4.2**

$e^{\lambda t}$ is exponentially stable, that is to say, there exist constants $M > 1$, $\omega > 0$ such that

$$\|e^{\lambda t}\| \leq Me^{-\omega t}$$

or equivalently

$$E(t) = \frac{1}{2} \int_0^1 \left[ \theta^2(x,t) + \sum_{i=1}^{N} w_i^2(x,t) \right] dx \leq Me^{-\omega t} E(0) = \frac{M}{2} e^{-\omega t} \int_0^1 \theta_0^2(x) dx \quad (82)$$

**Proof**

By the spectrum-determined growth condition claimed by Theorem 4.1, the asymptote for eigenvalues of (46) and (81), $e^{\lambda t}$ is exponentially stable if and only if

$$\Re \lambda < 0 \quad \forall \lambda \in \sigma(\mathcal{A})$$

However, this is a trivial fact due to strong dissipativity of (18), which shows that there is no eigenvalue of $\mathcal{A}$ located on the imaginary axis. The proof is complete.

Finally, we indicate by Lemma 3.1 that the $\omega$ in Theorem 4.2 satisfies

$$-\omega \geq \max\{-b_1, -b_2, \ldots, -b_N\}$$

This means that the decay rate of the energy cannot be less than the decay rate of the kernel. This interesting phenomena suggest us to impose the control to decrease the decay rate of the system, which, however, needs further investigation.
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