Boundary feedback stabilization and Riesz basis property of a 1-d first order hyperbolic linear system with $L^\infty$-coefficients

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\begin{abstract}
This paper deals with the boundary feedback stabilization problem of a wide class of linear first order hyperbolic systems with non-smooth coefficients. We propose static boundary inputs (actuators) which lead us to a closed loop system with non-smooth coefficients and non-homogeneous boundary conditions. Then, we prove the exponential stability of the closed loop system under suitable conditions on the coefficients and the feedback gains. The key idea of the proof is to combine the regularization techniques with the characteristics method. Furthermore, by the spectral analysis method, it is also shown that the closed loop system has a sequence of generalized eigenfunctions, which form a Riesz basis for the state space, and hence the spectrum-determined growth condition is deduced.
\end{abstract}

\section{Introduction}
Let us consider, in this paper, a general class of infinite-dimensional linear systems governed by first order hyperbolic linear equations with two independent variables

\begin{equation}
\begin{bmatrix}
\frac{\partial T_1(x,t)}{\partial t}
\frac{\partial T_2(x,t)}{\partial t}
\end{bmatrix}
=\begin{bmatrix}
K_1(x) & K_2(x)
\end{bmatrix}
\begin{bmatrix}
\frac{\partial T_1(x,t)}{\partial x}
\frac{\partial T_2(x,t)}{\partial x}
\end{bmatrix}
+\begin{bmatrix}
M_{11}(x) & M_{12}(x)
M_{21}(x) & M_{22}(x)
\end{bmatrix}
\begin{bmatrix}
T_1(x,t)
T_2(x,t)
\end{bmatrix},
\end{equation}

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\textsuperscript{1} The research of the author was supported by Sultan Qaboos University.
\textsuperscript{2} The research of the author was supported by the National Natural Science Foundation of China and the Program for New Century Excellent Talents in University of China.

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doi:10.1016/j.jde.2008.08.010
Throughout this article, the hypotheses stated below are assumed to be satisfied by the system equipped with the usual inner product.

The description of physical background of this system can be found in [5]. Indeed, as the reader may know, such systems arise in modeling the dynamics of tubular reactors, heat exchangers and packed gas absorbers which are frequently used in chemical engineering (see [5] and the references therein).

Now, we propose the feedback law

\begin{equation}
    u_1(t) = \varepsilon_1 T_1(0, t), \quad u_2(t) = \varepsilon_2 T_2(\ell, t),
\end{equation}

where \(\varepsilon_1\) and \(\varepsilon_2\) are positive feedback gains.

Henceforth, we use the following notations

\begin{equation}
    K(\cdot) = \begin{bmatrix}
        K_1(\cdot) & 0 \\
        0 & -K_2(\cdot)
    \end{bmatrix}, \quad M(\cdot) = \begin{bmatrix}
        M_{11}(\cdot) & M_{12}(\cdot) \\
        M_{21}(\cdot) & M_{22}(\cdot)
    \end{bmatrix}.
\end{equation}

Throughout this article, the hypotheses stated below are assumed to be satisfied by the system (1.1)–(1.4):

H.I (Non-smoothness) The coefficients \(K_i, M_{ij} \in L^\infty(0, \ell)\) for \(i, j = 1, 2\).

H.II (Strict hyperbolicity) There exists a constant \(\alpha > 0\) such that for any \(i = 1, 2\), we have \(K_i(x) \geq \alpha\) for almost every \(x \in (0, \ell)\).

H.III (Dissipativity)

H.IIIa For any \(\xi \in \mathbb{R}^2\), we have \(\xi^\top M(x) \xi \leq 0\) for almost every \(x \in (0, \ell)\).

H.IIIb The coefficient \(K_1(x)\) is almost everywhere monotone non-decreasing, that is, for each \(h > 0\), we have \(K_1(x + h) - K_1(x) \geq 0\) for almost every \(x \in (0, \ell)\) and \(K_2(x)\) is almost everywhere monotone non-increasing.

H.IIIc The feedback gains satisfy

\[0 < \varepsilon_1 \leq \sqrt{\frac{\alpha}{\|K_2\|_\infty}} \quad \text{and} \quad 0 < \varepsilon_2 \leq \sqrt{\frac{\alpha}{\|K_1\|_\infty}}.\]

It is worth mentioning that such assumptions H.I–H.II, H.IIIa and H.IIIb are satisfied by many chemical engineering systems like the counter-flow heat exchangers [5].

The aim of this work is to investigate, under the assumptions H.I–H.III, both stability and Riesz basis properties of the system (1.1)–(1.4).

The controllability and stabilizability problems of the system (1.1), with differentiable coefficients \(K_1(x)\) and \(K_2(x)\) and with various static boundary conditions, have been extensively studied in literature (see for instance [7,26–28,36] and the references therein). Later, the authors consider in [20] the system (1.1), with dynamic and static boundary conditions, and assume that \(K(\cdot)\) and \(M(\cdot)\) depend not only on the space variable \(x\) but also on the time variable \(t\). Under certain assumptions, several qualitative results are proved, such as well-posedness, asymptotic behavior characterization of spectrum for the system (with smooth coefficients \(K_i\) and \(M_{i,j}\)). In this case, where the coefficients are smooth,
it has been proved that the hypotheses H.I–H.II guarantee, in the state space \( \mathcal{H} \), the well-posedness of the problem (1.1)–(1.4) in the sense of semigroup theory (see [20] and [26]). Furthermore, the assumption H.III ensures the dissipativity of the system operator. Recently, the Riesz basis property has been investigated for the system (1.1), with a diagonal matrix \( M \) and dynamic as well as static boundary conditions [11,12]. As pointed out, the smoothness of the coefficients \( K_i \) and \( M_{ij} \) has been used in the works cited above. However, for dynamical processes met in chemical engineering such as heat exchangers and packed gas absorbers, the functions \( K_1(x) \) and \( K_2(x) \) may not be continuous [9]. This has motivated the work in [35] where the author considered a bilinear system (the coefficients \( K_i \) are time-varying) with homogeneous boundary conditions \((\varepsilon_1 = \varepsilon_2 = 0)\). Indeed, exact observability and exponential stability of such a system are proved in this article. Following this paper, Chentouf et al. [5] proved the exponential stability of (1.1)–(1.4) but with homogeneous boundary conditions. Actually, these two articles utilize the very known regularization (or mollification) idea and hence can be considered as a continuation of a vast literature dealing with existence and uniqueness of solutions of Cauchy problem for first order hyperbolic systems with discontinuous coefficients (see for instance [8,14,17,21–24] and the references therein). The same idea is used for the study of existence and uniqueness of solutions for a mixed initial–boundary value problem related to a particular form of first order hyperbolic systems with discontinuous coefficients [33]. More precisely, the author considered the \( n \)-dimensional system

\[
\frac{\partial}{\partial t} (AU) + \sum_{j=1}^{n} K_j^j \frac{\partial U}{\partial x_j} + BU = f,
\]

together with initial and conservative boundary conditions. Moreover, the matrix \( A \) has discontinuous coefficients but the matrices \( K_j^j \) have constant coefficients which is not the case for the system under consideration.

Note also that the system (1.1)–(1.4) with constant coefficients \( K_i \) and \( M_{ij} \) has been studied in [16]. In fact, using Huang’s result [13], the author proved that the system (1.1)–(1.4) with constant coefficients is exponentially stable and the spectrum-determined growth condition holds but in very special cases \( \varepsilon_1 = \frac{1}{\varepsilon_2} = \sqrt{\frac{K_1 M_{12}}{K_2 M_{22}}} \) and \( \varepsilon_1 = \sqrt{\frac{K_1 M_{12}}{K_2 M_{22}}} , \varepsilon_2 = 0 \) (see [16] for more details). This result has been improved in [37] in the sense that the Riesz basis property is shown to be true for the system (1.1)–(1.4) with constant coefficients for any \( \varepsilon_1 \varepsilon_2 \neq 0 \). Finally, similar physical systems have been the subject of many studies (stability, reachability, observability, controllability, . . . ) in [18,30–32].

The main contribution of this paper lies in: first, combining the classical regularization method with that of characteristics, we shall obtain, under the assumptions H.I–H.III, well-posedness and stability properties for the system (1.1)–(1.4) without any smoothness assumption on the system coefficients. This improves the stability results obtained in [5] where the boundary conditions are homogeneous and [16] where the coefficients are constants. Secondly, it is shown that the closed loop system has a sequence of generalized eigenfunctions, which form a Riesz basis for the state space, and hence the spectrum-determined growth condition holds. This generalizes the results proved in [11,12] where the matrix \( K \) (respectively the diagonal matrix \( M \)) has continuously differentiable (respectively continuous) coefficients and also those in [16] and the application in [37] where the coefficients are constants.

Now let us outline the content of this paper. In Section 2, we introduce the regularized system associated with the original one (1.1)–(1.4) and present its fundamental properties. In Section 3, we establish the uniform stability of the system (1.1)–(1.4). Results of this section have been partially announced in [6]. Finally, in the last section, we show that the Riesz basis property as well as the spectrum-determined growth condition hold for the system (1.1)–(1.4).

2. Regularization of the system (1.1)–(1.4)

The aim of this section is to introduce a system, with smooth coefficients, associated to the system (1.1)–(1.4). To this end, we shall use the well-known mollification method [1,19]. Indeed, the system (1.1)–(1.4) with non-smooth coefficients can be written as follows
where $T(t) = (T_1(t), T_2(t))^\top$, $\phi = (\phi_1, \phi_2)^\top$ and $A$ is an unbounded linear operator defined by

$$
\mathcal{D}(A) = \{(u, v)^\top \in H^1(0, \ell) \times H^1(0, \ell); \ v(0) = \varepsilon_1 u(0), \ u(\ell) = \varepsilon_2 v(\ell)\},
$$

(2.1)

and

$$
A = K \frac{\partial}{\partial x} + M.
$$

(2.2)

Now, given $g \in L^\infty(0, \ell)$ such that $0 < \alpha < g(x)$ for almost every $x \in (0, \ell)$, consider the function $\hat{g}$ as an extension of $g$ on $\mathbb{R}$ satisfying $\alpha \leq \hat{g}(x) \leq \|g\|_{\infty}, \ x \in \mathbb{R}$. Moreover, if $g$ is monotone on $(0, \ell)$, then $\hat{g}$ is piecewise monotone on $\mathbb{R}$ and obviously monotone on $[0, \ell]$.

Throughout this work, the mollification (or regularization) of $g$ will be denoted by $g^h$ ($h > 0$) and defined by

$$
g^h(x) = \int_{\mathbb{R}} \hat{g}(y) \omega_h(y - x) \, dy,
$$

where $\omega_h$ is the mollifier function [1,19].

Similarly, given the coefficients $K_i$ and $M_{ij}$, $i, j = 1, 2$, of the system (1.1)–(1.4), satisfying H.I–H.III, we shall consider the regularized system whose state variables are $T^h_1$ and $T^h_2$ and represent the solutions corresponding to the system (1.1)–(1.4) with regularized coefficients $K_i^h$ (resp. $M_{ij}^h$) associated to $K_i$ (resp. $M_{ij}$):

$$
\begin{bmatrix}
\frac{\partial T^h_1(x,t)}{\partial x}
\\
\frac{\partial T^h_2(x,t)}{\partial x}
\end{bmatrix} =
\begin{bmatrix}
K^h_i(x) \frac{\partial T^h_1(x,t)}{\partial x} \\
-K^h_i(x) \frac{\partial T^h_2(x,t)}{\partial x}
\end{bmatrix} +
\begin{bmatrix}
M^h_{11}(x) & M^h_{12}(x) \\
M^h_{21}(x) & M^h_{22}(x)
\end{bmatrix}
\begin{bmatrix}
T^h_1(x,t) \\
T^h_2(x,t)
\end{bmatrix}.
$$

(2.3)

\begin{align*}
T^h_2(0, t) &= \varepsilon_1 T^h_1(0, t) \quad \text{and} \quad T^h_1(\ell, t) = \varepsilon_2 T^h_2(\ell, t), \quad (2.4) \\
T^h_1(x, 0) &= \psi_1(x) \quad \text{and} \quad T^h_2(x, 0) = \psi_2(x), \quad (2.5)
\end{align*}

for $(x, t) \in (0, \ell) \times (0, +\infty)$.

The basic properties of the regularized coefficients $K_i^h$ (resp. $M_{ij}^h$) associated to $K_i$ (resp. $M_{ij}$) satisfying H.I–H.III are summarized in the following lemma whose proof is given in [5].

**Lemma 2.1.**

1. The coefficients $K_i^h$ and $M_{ij}^h$, $i, j = 1, 2$, of the system (2.3)–(2.5) are continuously differentiable and satisfy the hypotheses H.I–H.III.
2. For any $x \in \mathbb{R}$ and for $i, j = 1, 2$, we have

$$
\alpha \leq K_i^h(x) \leq \|K_i\|_{\infty} \quad \text{and} \quad \|M_{ij}^h(x)\|_{\infty} \leq \|M_{ij}\|_{\infty}, \ x \in \mathbb{R}.
$$

3. For all $1 \leq p < +\infty$ and for any constants $a, b$ with $0 \leq a < b \leq \ell$, we have

$$
\|K_i^h - K_i\|_{L^p(a,b)} \to 0 \quad \text{and} \quad \|M_{ij}^h - M_{ij}\|_{L^p(a,b)} \to 0, \quad i, j = 1, 2,
$$

as $h \to 0^+$. 


For any \(i, j = 1, 2\) and for any constants \(a, b\) with \(0 \leq a < b \leq \ell\), we have

\[
K^h_i \rightharpoonup K_i \quad \text{and} \quad M^h_{ij} \rightharpoonup M_{ij},
\]

in \(L^\infty(a, b)\) as \(h \to 0^+\).

We turn now to the formulation of the problem (2.3)–(2.5). For the sake of clarity, let

\[
K_h = \begin{bmatrix}
K^h_1 & 0 \\
0 & -K^h_2
\end{bmatrix}, \quad M_h = [M^h_{ij}], \quad \text{for } i, j = 1, 2.
\]

(2.6)

The system (2.3)–(2.5) can be written into the following abstract form

\[
\begin{cases}
T^h_t(t) = A_h T^h_t(t), \\
T^h(0) = \psi,
\end{cases}
\]

(2.7)

where

\[
T^h = (T^h_1, T^h_2)^T, \quad \psi = (\psi_1, \psi_2)^T \quad \text{and} \quad A_h \text{ is an unbounded linear operator defined by}
\]

\[
D(A_h) = \{(T^h_1, T^h_2)^T \in H^1(0, \ell) \times H^1(0, \ell); \ T^h_2(0) = \varepsilon_1 T^h_1(0), \ T^h_1(\ell) = \varepsilon_2 T^h_2(\ell)\},
\]

(2.8)

and

\[
A_h = K_h \frac{\partial}{\partial x} + M_h.
\]

(2.9)

**Lemma 2.2.** The operator \(A_h\) defined by (2.8)–(2.9) is \(m\)-dissipative in \(\mathcal{H}\).

**Proof.** Let \(T^h = (T^h_1, T^h_2)^T \in D(A_h)\). A direct computation yields

\[
\langle A_h T^h, T^h \rangle = \frac{1}{2} \left[ (\varepsilon_1^2 K^h_2(0) - K^h_1(0))(T^h_1(0))^2 + (\varepsilon_1^2 K^h_1(\ell) - K^h_2(\ell))(T^h_2(\ell))^2 \right] \\
+ \int_0^\ell \left[ -K^h_{1x}(x)(T^h_1(x))^2 + K^h_{2x}(x)(T^h_2(x))^2 \right] dx \\
+ \int_0^\ell \langle M_h(x)T^h(x), T^h(x) \rangle_{L^2} dx,
\]

where the subscript \(x\) stands for the derivative with respect to \(x\). Recall that \(K_{ix}^h\) exists, for \(i = 1, 2\), by part (1) of Lemma 2.1. This, together with Lemma 2.1 (parts (1) and (2)), implies that

\[
\langle A_h T^h, T^h \rangle = \frac{1}{2} \left[ (\varepsilon_1^2 \|K_2\|_\infty - \alpha)(T^h_1(0))^2 + (\varepsilon_2^2 \|K_1\|_\infty - \alpha)(T^h_2(\ell))^2 \right] \leq 0.
\]

(2.10)

Next, it is easy to check that the operator \(K_h \frac{\partial}{\partial x}\) with domain \(D(A_h)\) is maximal dissipative and so is the operator \(A_h = K_h \frac{\partial}{\partial x} + M_h\) since the matrix operator \(M_h\) is dissipative and bounded in \(\mathcal{H}\) [25]. □

**Remark 2.1.** (1) As mentioned in the introduction, the assumption H.III is needed only for the dissipativity of the system operator. In fact, the hypotheses H.I–H.II guarantee the well-posedness of the problem related to the system (2.3)–(2.5) (see [20,26]).

(2) The operator \(K_h \frac{\partial}{\partial x}\) with domain \(D(K_h) = D(A_h)\) generates a \(C_0\)-semigroup of contractions \(W_{K_h}(t)\) on \(\mathcal{H}\).

The following result is a direct consequence of semigroups theory [25].
Proposition 2.1.

(1) The operator $A_h$ defined by (2.8)–(2.9) is a generator of a $C_0$-semigroup of contractions $U_{K_h,M_h}(t)$ on $\mathcal{H}$, given by the variation of constant formula

$$U_{K_h,M_h}(t)\psi = W_{K_h}(t)\psi + \int_0^t W_{K_h}(t-\tau)M_hU_{K_h,M_h}(\tau)\psi\,d\tau,$$

for any $\psi \in \mathcal{H}$.

(2) For any $\psi \in D_\infty = \bigcap_{n=1}^{+\infty} D(A^n_h)$, the solution $T^h(t)$ of the regularized system (2.3)–(2.5) belongs to $D_\infty$ and $T^h(x,t) = (U_{K_h,M_h}(t)\psi)(x) \in C^\infty([0,T] \times [0,\ell])$, for any $T > 0$.

3. Stability results

3.1. Stability of the regularized system (2.3)–(2.5)

First, let us define a linear observation map

$$Y: D(A_h) \subset \mathcal{H} \mapsto \mathbb{R}^2 \quad \text{such that} \quad Y(u,v) = (u(0), v(\ell)). \quad (3.1)$$

Then, given an initial condition $\psi = (\psi_1, \psi_2)^T \in D(A_h)$, let $T^h = (T^h_1, T^h_2)^T$ be the corresponding solution of the regularized system (2.3)–(2.5) associated to the system (1.1)–(1.3) with input (1.4) and output (3.1). The aim of this subsection is to show that the solution $T^h = (T^h_1, T^h_2)^T$ satisfies the “observability” inequality [28,29]

$$\sigma \int_0^{T_0} \left[ ((T^h_1(0,t))^2 + (T^h_2(\ell,t))^2) \right] dt \geq \|\psi\|^2,$$

where $\sigma$ and $T_0$ are two positive constants independent of the parameter $h$. Next, the exponential stability of the regularized system (2.3)–(2.5) will be deduced. To our knowledge, this idea is due to Russell and has been used for many systems [4,5,27,28,35]. Also note that the exponential stability of such systems can be obtained directly from a very general theorem stated in [26].

Consider the characteristic curves $C_1$ and $C_2$ of the system (2.3)–(2.5)

$$C_1: \begin{cases} \dot{x}_1(t) = -K^h_i(x_1), \\ x_1(0) = \ell, \end{cases} \quad C_2: \begin{cases} \dot{x}_2(t) = K^h_i(x_2), \\ x_2(t_1) = 0, \end{cases}$$

where $t_1$ satisfies $x_1(t_1) = 0$. Moreover, we choose $t_2$ such that $x_2(t_2) = \ell$ (see Fig. 1). Since $K^h_i(x) \geq \alpha > 0$, $x \in \mathbb{R}$, $i = 1,2$, one can claim that the function $x_i$ is $C^1$-diffeomorphism on $(0,t_i)$, for $i = 1,2$. Hence, for $i = 1,2$, the inverse of $x_i$ exists and is denoted by $x_i^{-1}$. Furthermore, it is easy to show that $t_i \leq \ell/\alpha$, for $i = 1,2$.

In the sequel, $\Delta(IJK)$ denotes the surface delimited by the curves joining the points $I$, $J$ and $K$ whereas $\partial(\Delta IJK)$ denotes the boundary of $\Delta(IJK)$. From Proposition 2.1, it follows that for each $\psi \in D_\infty$, the corresponding solution belongs to $C^\infty(\Delta(IJK))$ and hence one can apply Green’s formula.

We have the following result (see [5,35] for similar arguments):

**Lemma 3.1.** The solution $T^h(t)$ of (2.3)–(2.5) stemmed from the initial data $\psi \in D_\infty$ satisfies the following inequalities
Fig. 1. Characteristics of the system (2.3)–(2.5).

\[
\int_0^\ell \left[ \frac{\alpha}{K_1^h(x)} \psi_1^2(x) + \frac{\|K_2\|_\infty}{K_2^h(x)} \psi_2^2(x) \right] dx \\
\leq e^{\gamma_1 t_1} \int_0^{t_1} \left[ (\alpha - e_1^2 \|K_2\|_\infty) (T_1^h(0, t))^2 + \left( 1 + \frac{K_1^h}{K_2^h}(x_1(t)) \right) \|K_2\|_\infty (T_2^h(x_1(t), t))^2 \right] dt 
\]

(3.2)

and

\[
\int_0^{t_2} \left[ \|K_1\|_\infty (T_1^h(\ell, t))^2 + \|K_2\|_\infty (T_2^h(\ell, t))^2 \right] dt \\
\geq e^{-\gamma_2 t} \|K_2\|_\infty \int_0^{t_1} \left( 1 + \frac{K_1^h}{K_2^h}(x_1(t)) \right) (T_2^h(x_1(t), t))^2 dt,
\]

(3.3)

where

\[
\gamma_1 = \|M_{12}\|_\infty + \|M_{21}\|_\infty + 2 \max \{ \|M_{11}\|_\infty; \|M_{22}\|_\infty \},
\]

\[
\gamma_2 = \frac{1}{\alpha} \left( \sqrt{\frac{\|K_1\|_\infty}{\|K_2\|_\infty}} \|M_{12}\|_\infty + \sqrt{\frac{\|K_2\|_\infty}{\|K_1\|_\infty}} \|M_{21}\|_\infty + 2 \max \{ \|M_{11}\|_\infty; \|M_{22}\|_\infty \} \right).
\]

**Proof.** We shall first prove the inequality (3.2). To do so, one can easily check that Eq. (2.3) yields

\[
\frac{\partial}{\partial t} \left[ \frac{\alpha}{K_1^h(x)} (T_1^h(x, t))^2 \right] = \frac{\partial}{\partial x} \left[ \alpha (T_1^h(x, t))^2 \right] \\
+ 2\alpha \left[ \frac{M_{11}^h(x)}{K_1^h(x)} (T_1^h(x, t))^2 + \frac{M_{12}^h(x)}{K_1^h(x)} T_1^h(x, t) T_2^h(x, t) \right],
\]
\[
\frac{\partial}{\partial t} \left[ \| K_2 \|_\infty \left( T_2^h(x, t) \right)^2 \right] = -\frac{\partial}{\partial x} \left[ \| K_2 \|_\infty \left( T_2^h(x, t) \right)^2 \right] \\
+ 2 \| K_2 \|_\infty \left[ \frac{M_{22}^h(x)}{K_2^h} (T_2^h(x, t))^2 + \frac{M_{21}^h(x)}{K_2^h} T_1^h(x, t) T_2^h(x, t) \right].
\]

Next, we add the above equations and integrate the sum over the domain \( \Delta(ABC) \) (see Fig. 1). Then, we apply Green formula to get after a careful computation

\[
M_t(\tau) \geq -\| K_2 \|_\infty \left( 1 + \frac{K_1^h N_2^h}{K_2^h} (x_1(\tau)) \right) \left( T_2^h(x_1(\tau), \tau) \right)^2 - (\alpha - \| K_2 \|_\infty \epsilon_1^2) \left( T_1^h(0, \tau) \right)^2 - \gamma_1 M(\tau),
\]

where

\[
M(\tau) = \int_0^{x_1(\tau)} \left[ \frac{\alpha}{K_1^h(x)} (T_1^h(x, t))^2 + \frac{\| K_2 \|_\infty}{K_2^h(x)} (T_2^h(x, t))^2 \right] dx.
\]

Finally, solving this differential inequality, one can deduce the desired inequality (3.2).

In order to obtain the second inequality (3.3), we use again Eq. (2.3). A straightforward computation gives

\[
\frac{\partial}{\partial t} \left[ \| K_1 \|_\infty \left( T_1^h(x, t) \right)^2 \right] = \frac{\partial}{\partial x} \left[ \| K_1 \|_\infty (T_1^h(x, t))^2 \right] \\
+ 2 \| K_1 \|_\infty \left[ \frac{M_{11}^h(x)}{K_1^h} (T_1^h(x, t))^2 + \frac{M_{12}^h(x)}{K_1^h} T_1^h(x, t) T_2^h(x, t) \right],
\]

\[
\frac{\partial}{\partial t} \left[ \| K_2 \|_\infty \left( T_2^h(x, t) \right)^2 \right] = -\frac{\partial}{\partial x} \left[ \| K_2 \|_\infty (T_2^h(x, t))^2 \right] \\
+ 2 \| K_2 \|_\infty \left[ \frac{M_{22}^h(x)}{K_2^h} (T_2^h(x, t))^2 + \frac{M_{21}^h(x)}{K_2^h} T_1^h(x, t) T_2^h(x, t) \right].
\]

Then, we subtract these two equations and consider the domain \( \Delta(CDE) \). Proceeding as for the inequality (3.2), one can obtain the required result. \( \square \)

Our first main result is:

**Theorem 3.1.** For any \( \psi \in \mathcal{H} \), we have:

1. The system (2.3)–(2.5) is \( T_0 \)-observable, where \( T_0 = \ell/\alpha \).
2. The system (2.3)–(2.5) is exponentially stable.

**Proof.** (1) By a standard argument of density of \( D_\infty \) in \( \mathcal{H} \) and the contraction of the semigroup \( U_{K_a, M_b}(t) \), it suffices to prove the theorem for any initial data \( \psi \in D_\infty \).

First, combining (2.4) with the inequalities (3.2) and (3.3) and using the fact that \( t_i \leq T_0 = \ell/\alpha \) for \( i = 1, 2 \), it follows that
\[
\int_0^\ell \left[ \frac{\alpha}{K_1(x)} \psi_1^2(x) + \frac{\|K_2\|_\infty}{K_2^2(x)} \psi_2^2(x) \right] dx 
\leq e^{\gamma_1 T_0 + \gamma_2 \ell} \int_0^{T_0} \left[ (\alpha - \varepsilon_1^2 K_2(x))(T^h_1(0, t))^2 + (\|K_2\|_\infty + \varepsilon_2^2 K_1(x))(T^h_2(\ell, t))^2 \right] dt.
\]

This implies that
\[
\|\psi\|^2 = \int_0^\ell \left[ \psi_1^2(x) + \psi_2^2(x) \right] dx \leq \sigma \int_0^{T_0} \left[ (T^h_1(0, t))^2 + (T^h_2(\ell, t))^2 \right] dt,
\] (3.4)

where
\[
\sigma = \frac{\|K_1\|_\infty}{\alpha} e^{\gamma_1 T_0 + \gamma_2 \ell} \max \{ \|K_1\|_\infty \varepsilon_2^2 + \|K_2\|_\infty ; \alpha - \varepsilon_1^2 \|K_2\|_\infty \}.
\]

Therefore, the system (2.3)–(2.5) is \(T_0\)-observable.

(2) It is important to note that \(\sigma > \max \{ \alpha - \varepsilon_1^2 \|K_2\|_\infty ; \alpha - \varepsilon_2^2 \|K_1\|_\infty \} > 0\). This leads us to claim that the proof of assertion (2) is a direct consequence of (2.10), (3.4) and the contractions of the semigroup \(U_{K_h, M_h}(t)\) (see [5] and [35] for similar situations).

3.2. Stability of the system (1.1)–(1.4)

In this subsection, we shall prove the exponential stability of the system (1.1)–(1.4). To do so, we will proceed by steps.

Let \(\alpha\) and \(\beta\) be two positive constants. We define as in [5,35]
\[
S^\infty_{\alpha, \beta}(0, \ell) = \{ f \in L^\infty(0, \ell); \alpha \leq f(x) \leq \beta \text{ almost everywhere} \}
\]
and
\[
C^\infty \cap S^\infty_{\alpha, \beta}(0, \ell) = \{ f \in S^\infty_{\alpha, \beta}(0, \ell); \text{ f is continuously differentiable} \}.
\]

The proof of the lemma below can be found in [3] (see also [35]):

**Lemma 3.2.** A bounded subset of \(L^\infty(0, \ell)\) is precompact with respect to the weak* topology. In particular, the set \(S^\infty_{\alpha, \beta}(0, \ell)\) is precompact with respect to the weak* topology. Furthermore, the subset \(C^\infty \cap S^\infty_{\alpha, \beta}(0, \ell)\) is dense in \(S^\infty_{\alpha, \beta}(0, \ell)\) with respect to the weak* topology.

Before stating the next result, let us refer the reader to our notations in (1.5) and (2.6).

**Proposition 3.1.** Given \(K_1, K_2 \in S^\infty_{\alpha, \beta}(0, \ell)\) satisfying H.II and H.III, there exists a unique \(C_0\)-semigroup of contractions \(W_K(t)\) in \(\mathcal{H}\) generated by the operator \(K \frac{\alpha}{\gamma_2} \). Moreover, for any \(\phi \in \mathcal{H}\), the application \(K \rightarrow W_K(\cdot)\phi\) is continuous with respect to the weak* topology of \(S^\infty_{\alpha, \beta}(0, \ell)\) and the uniform topology of \(C((0, T); \mathcal{H})\), for \(T > 0\).
Proof. Let \( K_i \in S_{\alpha,\beta}^{\infty}(0, \ell) \), for \( i = 1, 2 \), satisfying H.II–H.III. Combining Lemmas 2.1 and 3.2, one can claim the existence of a sequence \( K^n_i \in C^{\infty} \cap S_{\alpha,\beta}^{\infty}(0, \ell) \), satisfying H.II–H.III, such that for each \( i = 1, 2 \), we have

\[
K^n_i \rightharpoonup K_i, \quad \text{in } L^{\infty}(0, \ell), \quad \text{as } n \to +\infty.
\]

Using the second part of Remark 2.1, we conclude that there exists a \( C_0 \)-semigroup of contractions \( W_{K^n}(t) \), in \( H \), generated by the operator \( K \partial \partial x \). Next, following the same arguments used in [5] and [35], one can show that for any \( T > 0 \), the sequence of operators \( W_{K^n}(t) \) converges strongly in \( H \) uniformly on \((0, T)\) towards a limit operator \( W_K(t) = \lim_{n \to +\infty} W_{K^n}(t) \) which is a \( C_0 \)-semigroup of contractions in \( H \) associated to the operator \( K \partial \partial x \) with domain given by (2.2). This clearly implies the continuity of the mapping \( K \to W_K(\cdot)\phi \).

Given \( K_i \) and \( M_{ij} \) in \( S_{\alpha,\beta}^{\infty}(0, \ell) \) satisfying H.I–H.III, one can construct the \( C_0 \)-semigroup of contractions \( U_{K,M}(t) \) in \( H \) generated by the operator \( A = K \partial \partial x + M \) (see (2.1)–(2.2)) by means of the variations of constant formula

\[
U_{K,M}(t)\phi = W_K(t)\phi + \int_0^t W_K(t - \tau)MU_{K,M}(\tau)\phi \, d\tau,
\]

where \( W_K(t) \) represents the \( C_0 \)-semigroup of contractions obtained in the lemma above.

On the other hand, the mollifications functions \( K^h_i \) and \( M^h_{ij} \) obtained from \( K_i \) and \( M_i \) respectively and satisfying Lemma 2.1 lead us to define the semigroup of contractions \( U_{K^h,M^h}(t) \) (see Proposition 2.1). Moreover, we claim that for \( T > 0 \),

\[
U_{K^h,M^h}(t) \text{ converges strongly in } C((0, T), H) \text{ to } U_{K,M}(t), \quad \text{as } h \to 0^+.
\]

(P)

Indeed, let \( 0 \leq t \leq T \) and \( \phi \in D_\infty \). We have

\[
U_{K^h,M^h}(t)\phi - U_{K,M}(t)\phi = \int_0^t W_{K^h}(t - s)M^h_{ij} [U_{K^h,M^h}(s)\phi - U_{K,M}(s)\phi] \, ds
\]

\[
+ \int_0^t \left[ W_{K^h}(t - s) - W_K(t - s) \right] MU_{K,M}(s)\phi \, ds
\]

\[
+ \int_0^t W_{K^h}(t - s)(M^h - M)U_{K,M}(s)\phi \, ds
\]

\[
+ W_{K^h}(t)\phi - W_K(t)\phi. \tag{3.5}
\]

This, together with contractions property of the semigroup \( W_{K^h}(t) \), implies that

\[
\psi_h(t) \leq \int_0^t \|M^h\| \psi_h(s) \, ds + \int_0^t \| (M^h - M)U_{K,M}(s)\phi \| \, ds + \frac{\| W_{K^h}(t)\phi - W_K(t)\phi \|}{\epsilon^2_1},
\]

where \( \epsilon^h_1 \) is a constant depending on \( h \).
\[
\begin{align*}
\psi \big|_{\Theta^b} &+ \int_0^t \big[ W_{K_h}(t-s) - W_K(t-s) \big] M U_{K,M}(s) \phi \, ds,
\end{align*}
\]
where \(\psi(t) = \|U_{K_h,M_h}(t)\phi - U_{K,M}(t)\phi\|\). Using Gronwall’s inequality and the density of \(D_{\infty}\) in \(H\), it follows that our claim holds if the three last terms in the right side of the integral inequality (3.6) tend to zero as \(h \to 0^+\).

Firstly, consider the term \(\Theta^b\). We know by part (3) of Lemma 2.1 that, for \(i, j = 1, 2\), \(\|M^b_{ij} - M_{ij}\|_{L^p(0,t)} \to 0^+\). Hence applying Hölder inequality and using the contraction of the semigroup \(U_{K,M}(t)\) yield \(\|(M_h - M)U_{K,M}(t)\phi\| \to 0\), for \(t > 0\). Then, one can apply Lebesgue convergence theorem to obtain the convergence of \(\Theta^b\) to zero as \(h \to 0^+\).

Secondly, the second term \(\Theta^b\) goes to zero as \(h \to 0^+\) by means of Proposition 3.1. Finally, the convergence of the third term \(\Theta^b\) can be obtained as in the proof of Lemma 8 in [5].

Our second main result is:

**Theorem 3.2.** The system (1.1)–(1.4) is exponentially stable in \(H\).

**Proof.** Let \(\phi \in D_{\infty}\) and \(K\) and \(M\) be the matrices defined by (1.5) and satisfying H.I–H.III. It follows from Lemma 2.1 and Theorem 3.1 that there exist matrices \(K_h\) and \(M_h\) such that the associated semi-group \(U_{K_h,M_h}(t)\) is exponentially stable, that is, there exist positive constants \(M\) and \(\omega\) independent of \(h\) such that \(\|U_{K_h,M_h}(t)\phi\| \leq Me^{-\omega t}\|\phi\|\), for all \(\phi \in D_{\infty}\). Now, using the assertion (P), we obtain after taking the limit \(h \to 0^+\),

\[\|U_{K,M}(t)\phi\| \leq Me^{-\omega t}\|\phi\|\].

Finally, by a standard argument of density, the result is extended to \(\phi \in H\). \(\square\)

**4. Riesz basis generation**

For the sake of clarity and without loss of generality, let us consider the following variant of the system (1.1)–(1.4):

\[
\begin{align*}
\begin{bmatrix}
\frac{\partial T_1(x,t)}{\partial t} \\
\frac{\partial T_2(x,t)}{\partial t}
\end{bmatrix}
&= \begin{bmatrix}
K_1(x) \frac{\partial T_1(x,t)}{\partial x} - K_2(x) \frac{\partial T_2(x,t)}{\partial x} \\
- K_2(x) \frac{\partial T_1(x,t)}{\partial x} + K_1(x) \frac{\partial T_2(x,t)}{\partial x}
\end{bmatrix} + \begin{bmatrix}
K_1(x)M_{11}(x) & K_1(x)M_{12}(x) \\
K_2(x)M_{21}(x) & K_2(x)M_{22}(x)
\end{bmatrix}
\begin{bmatrix}
T_1(1,t) \\
T_2(2,t)
\end{bmatrix},
\end{align*}
\]

(4.1)

\[T_2(0,t) = \varepsilon_1 T_1(0,t) \quad \text{and} \quad T_1(1,t) = \varepsilon_2 T_2(1,t),\]  

(4.2)

\[T_1(x,0) = \phi_1(x) \quad \text{and} \quad T_2(x,0) = \phi_2(x),\]  

(4.3)

for \((x,t) \in (0,1) \times (0, +\infty)\). Here the coefficients \(K_i\) and \(M_{ij}\), \(i, j = 1, 2\), of the above system are assumed to satisfy only the hypotheses H.I–H.II.

Clearly, the regularized system associated to the system (4.1)–(4.3) is given by the following variant of the system (2.3)–(2.5), namely,

\[
\begin{align*}
\begin{bmatrix}
\frac{\partial T_1^h(x,t)}{\partial t} \\
\frac{\partial T_2^h(x,t)}{\partial t}
\end{bmatrix}
&= \begin{bmatrix}
K_1^h(x) \frac{\partial T_1^h(x,t)}{\partial x} - K_2^h(x) \frac{\partial T_2^h(x,t)}{\partial x} \\
- K_2^h(x) \frac{\partial T_1^h(x,t)}{\partial x} + K_1^h(x) \frac{\partial T_2^h(x,t)}{\partial x}
\end{bmatrix} + \begin{bmatrix}
K_1^h(x)M_{11}^h(x) & K_1^h(x)M_{12}^h(x) \\
K_2^h(x)M_{21}^h(x) & K_2^h(x)M_{22}^h(x)
\end{bmatrix}
\begin{bmatrix}
T_1^h(1,t) \\
T_2^h(2,t)
\end{bmatrix},
\end{align*}
\]

(4.4)

\[T_2^h(0,t) = \varepsilon_1 T_1^h(0,t) \quad \text{and} \quad T_1^h(1,t) = \varepsilon_2 T_2^h(1,t),\]  

(4.5)

\[T_1^h(x,0) = \psi_1(x) \quad \text{and} \quad T_2^h(x,0) = \psi_2(x),\]  

(4.6)

for \((x,t) \in (0,1) \times (0, +\infty)\).
In this section, we shall show that the system (4.1)–(4.3) has a set of generalized eigenfunctions which form a Riesz basis in $\mathcal{H}$.

### 4.1. Riesz basis property for the regularized system (4.4)–(4.6)

Let us investigate, in this subsection, the Riesz basis generation for the regularized system (4.4)–(4.6). The key idea is to apply the following result which provides a useful way to verify the Riesz basis property for the generalized eigenvectors of a linear operator with compact resolvent in a Hilbert space.

**Theorem 4.1.** (See [10].) (See also [15].) Let $A$ be a densely defined discrete operator (i.e., there is a $\lambda^h \in \rho(A)$, the resolvent of $A$, such that $(\lambda^h I - A)^{-1}$ is compact on $H$) in a Hilbert space $H$. Let $\{\Phi_n\}_{n=1}^{\infty}$ be a Riesz basis for $H$. If there are an integer $N \geq 0$ and a sequence of generalized eigenvectors $\{\Psi_n\}_{n=1}^{N}$ of $A$ such that

$$\sum_{n=N+1}^{\infty} \|\Phi_n - \Psi_n\|^2 < \infty,$$

then

1. there are integer $M > N$ and generalized eigenvectors $\{\Psi_{nM}\}_{n=1}^{M}$ of $A$ such that $\{\Psi_{nM}\}_{n=1}^{M} \cup \{\Psi_{n}^{\infty}\}_{n=M+1}^{\infty}$ form a Riesz basis for $H$;
2. if $\{\Psi_{nM}\}_{n=1}^{M} \cup \{\Psi_{n}^{\infty}\}_{n=M+1}^{\infty}$ are the generalized eigenvectors corresponding to eigenvalues $\{\sigma_n\}_{n=1}^{\infty}$ of $A$, then $\sigma(A) = \{\sigma_n\}_{n=1}^{\infty}$ where $\sigma_n$ is accounted according to its algebraic multiplicity;
3. if there is an integer $M_0 > 0$ such that $\sigma_n \neq \sigma_m$ for all $m, n > M_0$, then there is an integer $N_0 > M_0$ such that all $\sigma_n$ are algebraically simple for all $n > N_0$.

First, the system (4.4)–(4.6) can be written in $\mathcal{H}$ as follows

$$T^h(t) = A^h T^h(t), \quad T^h(0) = \psi,$$

where $T^h(t) = (T^h_1(\cdot, t), T^h_2(\cdot, t))^T$, $\psi = (\psi_1, \psi_2)^T$ and $A^h$ is an unbounded linear operator defined by

$$D(A^h) = \{(u^h, v^h)^T \in H^1(0, 1) \times H^1(0, 1); \ v^h(0) = \varepsilon_1 u^h(0), \ u^h(1) = \varepsilon_2 v^h(1)\},$$

and

$$A^h = K_h \frac{\partial}{\partial x} + \begin{bmatrix} K^h_{11} M^h_1 & K^h_{11} M^h_{12} \\ K^h_{21} M^h_1 & K^h_{22} M^h_{22} \end{bmatrix}. \tag{4.8}$$

The eigenvalue problem of (4.4)–(4.6) is

$$\lambda^h \begin{bmatrix} T^h_1(x) \\ T^h_2(x) \end{bmatrix} = \begin{bmatrix} K^h_{11}(x) \frac{d^2}{dx^2} (x) \\ -K^h_{22}(x) \frac{d^2}{dx^2} (x) \end{bmatrix} + \begin{bmatrix} K^h_{11}(x) M^h_{11}(x) & K^h_{11}(x) M^h_{12}(x) \\ K^h_{21}(x) M^h_{11}(x) & K^h_{22}(x) M^h_{22}(x) \end{bmatrix} \begin{bmatrix} T^h_1(x) \\ T^h_2(x) \end{bmatrix}, \tag{4.9}$$

and

$$T^h_2(0) = \varepsilon_1 T^h_1(0), \quad T^h_1(1) = \varepsilon_2 T^h_2(1). \tag{4.10}$$

Using the notations in (2.6), the problem (4.9) changes

$$T^h_\lambda(x) = \lambda^h G^h(x) T^h(x) - \bar{T} M^h(x) T^h(x),$$

where
where \( T^h(x) = [T^h_1(x), T^h_2(x)]^\top \), the subscript \( x \) stands for the derivative with respect to \( x \) and
\[
G_h = K_h^{-1} = \begin{bmatrix} G^h_1 & 0 \\
0 & -G^h_2 \end{bmatrix}, \quad G^h_1 = 1/K^h_1, \quad G^h_2 = 1/K^h_2, \quad \tilde{T} = \begin{bmatrix} 1 & 0 \\
0 & -1 \end{bmatrix}. \tag{4.12}
\]
Moreover, the boundary conditions (4.10) can be formulated as follows
\[
W^0 T^h(0) + W^1 T^h(1) = 0, \tag{4.13}
\]
where
\[
W^0 = \begin{bmatrix} \epsilon_1 & -1 \\
0 & 0 \end{bmatrix}, \quad W^1 = \begin{bmatrix} 0 & 0 \\
1 & -\epsilon_2 \end{bmatrix}.
\]

We have the following result.

**Theorem 4.2.** Let \( \lambda^h \in \mathbb{C} \setminus \{0\} \). For \( x \in [0, 1] \), set
\[
E^h_1(x) := \int_0^x G^h_1(\xi) \, d\xi, \quad E^h_2(x) := \int_0^x G^h_2(\xi) \, d\xi \tag{4.14}
\]
and
\[
E^h(x, \lambda^h) := \begin{bmatrix} \exp(\lambda^h E^h_1(x)) & 0 \\
0 & \exp(-\lambda^h E^h_2(x)) \end{bmatrix}. \tag{4.15}
\]
Then there exists a fundamental matrix solution \( \hat{G}^h(x, \lambda^h) \) for the system (4.11), such that for large enough \( |\lambda^h| \),
\[
\hat{G}^h(x, \lambda^h) = (\hat{G}^h_0(\cdot) + \mathcal{O}((\lambda^h)^{-1})) E^h(\cdot, \lambda^h), \tag{4.16}
\]
where
\[
\hat{G}^h_0(x) := \text{diag}[C^h_1(x), C^h_2(x)] \tag{4.17}
\]
and
\[
C^h_1(x) := \exp\left(-\int_0^x M^h_{11}(\xi) \, d\xi \right), \quad C^h_2(x) := \exp\left(\int_0^x M^h_{22}(\xi) \, d\xi \right). \tag{4.18}
\]

**Proof.** First, using (4.11)–(4.12), one can easily check that Assumption 2.1 of [34] is satisfied and hence a direct application of Theorem 2.2 in [34] (see also [2]) shows that a fundamental matrix solution of (4.11) is of the following form
\[
\hat{G}^h(x, \lambda^h) = (\hat{G}^h_0(x) + (\lambda^h)^{-1} \hat{G}^h_1(x) + (\lambda^h)^{-2} \Theta^h(x, \lambda^h)) E^h(x, \lambda^h), \tag{4.19}
\]
where \( \Theta^h(x, \lambda^h) \) is uniformly bounded in \( \lambda^h \) and \( x \in [0, 1] \). Since \( G^h(\cdot) \), given by (4.12), is a diagonal matrix, we conclude that \( E^h(\cdot, \lambda^h) \) defined in (4.14) and (4.15) is a fundamental matrix solution for the leading term equation
\[
T^h_\chi(x) - \lambda^h G^h(x) T^h(x) = 0.
\]
That is, we have

\[ E^h(x, \lambda^h) = \lambda^h G^h(x) E^h(x, \lambda^h). \]

Moreover, substituting (4.19) into (4.11) leads us to obtain the left-hand side of (4.11)

\[ \hat{G}^h_{\Delta}(x, \lambda^h) = \left( \hat{G}^h_0(x) + \frac{1}{\lambda^h} \hat{G}^h_1(x) + (\lambda^h)^{-2} \phi^h(x, \lambda^h) \right) E^h(x, \lambda^h) \]

\[ + \lambda^h \left( \hat{G}^h_0(x) + \frac{1}{\lambda^h} \hat{G}^h_1(x) + (\lambda^h)^{-2} \phi^h(x, \lambda^h) \right) G^h(x) E^h(x, \lambda^h), \]

as well as its right-hand side

\[ (\lambda^h c^h(x) - \tilde{T} M^h(x)) \left( \hat{G}^h_0(x) + \frac{1}{\lambda^h} \hat{G}^h_1(x) + (\lambda^h)^{-2} \phi^h(x, \lambda^h) \right) E^h(x, \lambda^h). \]

Then comparing their coefficients, we get (according to the coefficients of \( \lambda^1 \) and \( \lambda^0 \))

\[ \hat{G}^h_0(x) G^h(x) - G^h(x) \hat{G}^h_0(x) = 0 \]  

(4.20)

and

\[ \hat{G}^h_0(x) + \hat{G}^h_1(x) G^h(x) - G^h(x) \hat{G}^h_1(x) + \tilde{T} M^h(x) \hat{G}^h_0(x) = 0. \]  

(4.21)

Finally, it remains to show that the leading order term \( \hat{G}^h_0(x) \) is given by (4.17). In fact, from (4.20) and \( \hat{G}^h_1 \neq -\hat{G}^h_2 \) in (4.12), it follows that the matrix function \( \hat{G}^h_0(x) \) is of diagonal

\[ \hat{G}^h_0(x) := \text{diag}[g^h_{11}(x), g^h_{22}(x)] \]

and its entries can be obtained by substituting them into (4.21), that is,

\[ (g^h_{11})_x = -M^h_{11}(x) g^h_{11}, \quad (g^h_{22})_x = M^h_{22}(x) g^h_{22} \]  

(4.22)

with \( \hat{G}^h_0(0) = I \). Hence (4.17) follows. \( \Box \)

We are now in a position to estimate the asymptotics of the eigenvalues \( \lambda^h \). First, note that the eigenvalues \( \lambda^h \) of the first order linear system (4.11) and (4.13) can be obtained as the zeros of the characteristic determinant

\[ \Delta(\lambda^h) := \det[W^0 \tilde{G}^h(0, \lambda^h) + W^1 \tilde{G}^h(1, \lambda^h)], \quad \lambda^h \in \mathbb{C}. \]  

(4.23)

A simple computation gives

\[ \Delta(\lambda^h) = \det \begin{bmatrix} E^1 & -1 \\ (C^h_1(1) + O((\lambda^h)^{-1})) e^{\lambda^h E^h_1(1)} & -\varepsilon_2 (C^h_2(1) + O((\lambda^h)^{-1})) e^{-\lambda^h E^h_2(1)} \end{bmatrix} \]

(4.24)

and

\[ \Delta(\lambda^h) = (C^h_1(1) + O((\lambda^h)^{-1})) e^{\lambda^h E^h_1(1)} - \varepsilon_1 \varepsilon_2 (C^h_2(1) + O((\lambda^h)^{-1})) e^{-\lambda^h E^h_2(1)}, \]

(4.25)
where $E_i^h(x)$ and $C_i^h(x)$ ($i = 1, 2$) are given by (4.14) and (4.18), respectively. Hence

$$\Delta(\lambda^h) = C_1^h(1)e^{\lambda^h E_1^h(1)} - \varepsilon_1 \varepsilon_2 C_2^h(1)e^{-\lambda^h E_2^h(1)} + \mathcal{O}((\lambda^h)^{-1}).$$  \hspace{1cm} (4.26)

Applying Rouché’s theorem, the roots of (4.26) can be estimated by those of

$$C_1^h(1)e^{\lambda^h E_1^h(1)} - \varepsilon_1 \varepsilon_2 C_2^h(1)e^{-\lambda^h E_2^h(1)} = 0,$$

which, using (4.18), yields

$$e^{\lambda^h (E_1^h(1) + E_2^h(1))} = \frac{\varepsilon_1 \varepsilon_2 C_2^h(1)}{C_1^h(1)} = \varepsilon_1 \varepsilon_2 \exp\left(\int_0^1 [M_{11}^h(\xi) + M_{22}^h(\xi)] d\xi\right)$$

and

$$\lambda_n^h (C_1^h(1) + E_2^h(1)) = \ln(\varepsilon_1 \varepsilon_2) + \left(\int_0^1 [M_{11}^h(\xi) + M_{22}^h(\xi)] d\xi\right) + 2n\pi i, \quad n \in \mathbb{Z}.$$

Hence, the roots of (4.26) are

$$\lambda_n^h = \frac{\ln(\varepsilon_1 \varepsilon_2) + \left(\int_0^1 [M_{11}^h(\xi) + M_{22}^h(\xi)] d\xi\right) + 2n\pi i}{E_1^h(1) + E_2^h(1)} + \mathcal{O}(n^{-1})$$

where $n \in \mathbb{Z}$. The corresponding eigenfunction to $\lambda_n^h$ is

$$T_n^h(x) = \begin{bmatrix} \varepsilon_1 C_1^h(x) \exp(\lambda_n^h E_1^h(x)) + \mathcal{O}((\lambda_n^h)^{-1}) \\ \varepsilon_1 C_2^h(x) \exp(\lambda_n^h E_2^h(x)) + \mathcal{O}((\lambda_n^h)^{-1}) \\ \exp(\int_0^x [\lambda_n^h C_1^h(\xi) - M_{11}^h(\xi)] d\xi) + \mathcal{O}((\lambda_n^h)^{-1}) \\ \varepsilon_1 \exp(\int_0^x [\lambda_n^h C_2^h(\xi) - M_{22}^h(\xi)] d\xi) + \mathcal{O}((\lambda_n^h)^{-1}) \end{bmatrix}.$$  \hspace{1cm} (4.28)

Therefore, the eigenfunction to $\lambda_n^h$ given by (4.27) has the following asymptotic form

$$T_n^h(x) = H^h(x) \begin{bmatrix} \exp(2n\pi i E_1^h(x)) + \mathcal{O}(n^{-1}) \\ \exp(2n\pi i E_2^h(x)) + \mathcal{O}(n^{-1}) \end{bmatrix},$$  \hspace{1cm} (4.29)

where $E_i^h(x)$ are given by (4.14) and $H^h(x)$ is the following $2 \times 2$ matrix

$$H^h(x) = \begin{bmatrix} H_{11}^h(x) & 0 \\ 0 & \varepsilon_1 H_{22}^h(x) \end{bmatrix},$$  \hspace{1cm} (4.30)

with

$$H_{11}^h(x) = \exp\left(\frac{\ln(\varepsilon_1 \varepsilon_2) + \left(\int_0^1 [M_{11}^h(\xi) + M_{22}^h(\xi)] d\xi\right) E_1^h(x)}{E_1^h(1) + E_2^h(1)} - \int_0^x M_{11}^h(\xi) d\xi\right).$$  \hspace{1cm} (4.31)
and

\[ H_{22}^h(x) = \exp \left( \frac{\ln(\varepsilon_1 \varepsilon_2) + \int_0^1 [M_{11}^h(\xi) + M_{22}^h(\xi)] d\xi}{E_1^h(1) + E_2^h(1)} \right) - \int_0^x M_{22}^h(\xi) d\xi. \]  \hspace{1cm} (4.32)

It is easily seen that \( H^h(x) \) is invertible in \( x \) and

\[ |H^h(x)| = \varepsilon_1 H_{11}^h(x) H_{22}^h(x) \neq 0. \]

Let us summarize these results as follows.

**Theorem 4.3.** Let \( A^h \) be the operator defined by (4.7) and (4.8). The eigenvalue of \( A^h \) has the asymptotic expression

\[ \lambda_n^h = \frac{\ln(\varepsilon_1 \varepsilon_2) + \int_0^1 [M_{11}^h(\xi) + M_{22}^h(\xi)] d\xi + 2n\pi i}{\int_0^1 [C_1^h(\xi) + C_2^h(\xi)] d\xi} + O(n^{-1}). \]  \hspace{1cm} (4.33)

and its corresponding asymptotic eigenfunction is given by (4.29).

Now, define a linear operator \( A_0^h \) in \((L^2[0, 1])^2\) by

\[ A_0^h = \begin{bmatrix} K_1(x) & 0 \\ 0 & -K_2^h(x) \end{bmatrix} \frac{\partial}{\partial x} + \begin{bmatrix} K_1^h(x) M_{11}^h(x) & 0 \\ 0 & K_2(x) M_{22}^h(x) \end{bmatrix}, \]  \hspace{1cm} (4.34)

with

\[ D(A_0^h) = \{ (u_0^h, v_0^h)^\top \in (H^1(0, 1))^2; \ v_0^h(0) = \varepsilon_1 u_0^h(0), \ u_0^h(1) = \varepsilon_2 v_0^h(1) \}. \]  \hspace{1cm} (4.35)

By using Theorem 2.1, Corollary 2.1 and Theorem 2.6 of [12], it is seen that the operator \( A_0^h \) defined by (4.34) and (4.35) has the following properties:

(i) the operator \( A_0^h \) is a discrete operator, in other words, for any \( \lambda^h \in \rho(A_0^h) \), the resolvent operator \( R(\lambda^h, A_0^h) \) is compact on \((L^2[0, 1])^2\);

(ii) the operator \( A_0^h \) generates a \( C_0 \)-group in \((L^2[0, 1])^2\);

(iii) there is a set of generalized eigenfunctions of \( A_0^h \), which forms a Riesz basis with parentheses for \((L^2[0, 1])^2\);

(iv) the spectrum-determined growth condition holds, that is, \( S(A_0^h) = \omega(A_0^h) \), where

\[ S(A_0^h) = \sup \{ \Re(\lambda^h); \ \lambda^h \in \sigma(A_0^h) \} \]

and

\[ \omega(A_0^h) = \inf \{ \omega; \ \exists \mu \geq 1 \text{ such that } \| e^{A_0^h t} \| \leq \mu e^{\omega t} \text{ for all } t \geq 0 \}. \]

Consequently, we have the following theorem.

**Theorem 4.4.** The operator \( A_0^h \) defined by (4.34) and (4.35) has the asymptotic eigenvalues \( \lambda^h_{0n} \) and the corresponding eigenfunctions \( T^h_{0n} \), given by (4.33) and (4.29), respectively. Moreover, each eigenvalue is simple when its modulus is large enough. Hence, there is a set of generalized eigenfunctions of \( A_0^h \) which forms a Riesz basis in \((L^2[0, 1])^2\).
Next, we are going to state the third main result, namely, the Riesz basis generation of $A^h$ by applying the general results stated in Theorem 4.1.

**Theorem 4.5.** The generalized eigenfunctions of the operator $A^h$ defined by (4.7) and (4.8) form a Riesz basis in $(L^2[0, 1])^2$ and hence the spectrum-determined growth condition $\omega(A^h) = s(A^h)$ holds true for the $C_0$-semigroup generated by $A^h$.

**Proof.** Since $A^h$ and $A_0^h$ have the same asymptotic forms of eigenvalues and eigenfunctions, the Riesz basis property can be obtained directly from Theorem 4.1 and the following

$$\sum_{n \geq N} \left\| T_n^h - T_n^{h_0} \right\|^2 = \sum_{n \geq N} \mathcal{O}(n^{-2}) < \infty,$$

where $N$ is a large enough positive integer. Finally, one can check that for sufficiently large $n$, the eigenvalues $\lambda_n^h$ of $A^h$ are algebraically simple. This, together with the Riesz basis property, implies the spectrum-determined growth condition. \( \square \)

### 4.2. Riesz basis property for the system (4.1)–(4.3)

We turn now to the investigation of the Riesz basis generation for the system (4.1)–(4.3). The key idea, which has been used in Section 3.2, is to use the spectral analysis carried out for the regularized system (4.4)–(4.6) and its operator $A^h$ defined by (4.7) and (4.8).

The system (4.1)–(4.3) can be written in $\mathcal{H}$ as follows

$$T_t(t) = AT(t), \quad T(0) = \phi,$$

where $T(t) = (T_1(\cdot, t), T_2(\cdot, t))^T$, $\phi = (\phi_1, \phi_2)^T$ and $A$ is an unbounded linear operator defined by

$$D(A) = \{(u, v)^T \in H^1(0, 1) \times H^1(0, 1); \; v(0) = \varepsilon_1 u(0), \; u(1) = \varepsilon_2 v(1)\},$$

(4.36)

and

$$A = K \frac{\partial}{\partial x} + \begin{bmatrix} K_1 M_{11} & K_1 M_{12} \\ K_2 M_{21} & K_2 M_{22} \end{bmatrix}.$$  

(4.37)

Then, the eigenvalue problem associated to the system (4.1)–(4.3) is

$$\lambda \begin{bmatrix} T_1(x) \\ T_2(x) \end{bmatrix} = \begin{bmatrix} K_1(x) \frac{dT_1}{dx}(x) \\ -K_2(x) \frac{dT_2}{dx}(x) \end{bmatrix} + \begin{bmatrix} K_1(x) M_{11}(x) & K_1(x) M_{12}(x) \\ K_2(x) M_{21}(x) & K_2(x) M_{22}(x) \end{bmatrix} \begin{bmatrix} T_1(x) \\ T_2(x) \end{bmatrix},$$

(4.38)

and

$$T_2(0) = \varepsilon_1 T_1(0), \quad T_1(1) = \varepsilon_2 T_2(1).$$

(4.39)

**Theorem 4.6.** The generalized eigenfunctions of the operator $A$ defined by (4.36) and (4.37) form a Riesz basis in $(L^2[0, 1])^2$ and hence the spectrum-determined growth condition $\omega(A) = s(A)$ holds true for the $C_0$-semigroup generated by $A$. 
Proof. First, let

\[ E_1(x) := \int_0^x \frac{d\xi}{K_1(\xi)}, \quad E_2(x) := \int_0^x \frac{d\xi}{K_2(\xi)} \]

and

\[ H(x) = \begin{bmatrix} H_{11}(x) & 0 \\ 0 & \varepsilon_1 H_{22}(x) \end{bmatrix}, \]

where

\begin{align*}
H_{11}(x) &= \exp \left( \frac{\ln(\varepsilon_1 \varepsilon_2) + \int_0^1 [M_{11}(\xi) + M_{22}(\xi)] d\xi}{E_1(1) + E_2(1)} E_1(x) \right) - \int_0^x M_{11}(\xi) d\xi \\
H_{22}(x) &= \exp \left( \frac{\ln(\varepsilon_1 \varepsilon_2) + \int_0^1 [M_{11}(\xi) + M_{22}(\xi)] d\xi}{E_1(1) + E_2(1)} E_2(x) \right) - \int_0^x M_{22}(\xi) d\xi.
\end{align*}

Then, we define

\[ T_n(x) = H(x) \begin{bmatrix} \exp(\frac{2n\pi i E_1(x)}{E_1(1) + E_2(1)}) + O(n^{-1}) \\ \exp(\frac{2n\pi i E_2(x)}{E_1(1) + E_2(1)}) + O(n^{-1}) \end{bmatrix} \]

and

\[ \lambda_n = \frac{\ln(\varepsilon_1 \varepsilon_2) + \int_0^1 [M_{11}(\xi) + M_{22}(\xi)] d\xi + 2n\pi i}{\int_0^1 [K_1(\xi) + K_2(\xi)] d\xi} + O(n^{-1}). \]

By a straightforward computation, one can check that \( AT_n = \lambda_n T_n \) for \( n \) large enough, and hence the asymptotic eigenfunctions of the operator \( A \) corresponding to the eigenvalues are given respectively by (4.42) and (4.43). In turn, it is worth mentioning that checking the above statement does not need any smoothness condition on the coefficients. Next, using the estimates (4.29), (4.33), (4.42) and (4.43), we obtain after a careful calculation where Hölder inequality and Lebesgue convergence theorem are applied

\[ |\lambda_n^h - \lambda_n| \to 0 \quad \text{and} \quad \|T_n^h - T_n\| \to 0, \quad \text{as} \ h \to 0^+. \]

Finally, using Theorems 4.3 and 4.5, one can deduce the result. \( \square \)

Acknowledgments

The authors are grateful to the anonymous referee for his/her constructive criticism and valuable suggestions and for having mentioned to them Ref. [18]. The first author also acknowledges the support of Sultan Qaboos University.
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