A RIESZ BASIS METHODOLOGY FOR PROPORTIONAL AND INTEGRAL OUTPUT REGULATION OF A ONE-DIMENSIONAL DIFFUSIVE-WAVE EQUATION

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Abstract. In this article, we consider a dam-river system modeled by a diffusive-wave equation. This model is commonly used in hydraulic engineering to describe dynamic behavior of the unsteady flow in a river for shallow water when the flow variations are not important. In order to stabilize and regulate the system, we propose a proportional and integral boundary controller. Contrary to many physical systems, we end up with a nondissipative closed-loop system with noncollocated actuators and sensors. We show that the closed-loop system is a Riesz spectral system and generates an analytic semigroup. Then, we shall be able to assign the spectrum of the closed-loop system in the open left half-plane to ensure its exponential stability as well as the output regulation independently of any known or unknown constant perturbation. These results are illustrated by several numerical examples.

Key words. dam-river system, proportional and integral boundary control, analytic semigroup, Riesz basis, stability

AMS subject classifications. 35B37, 35K20, 35P10, 35P20, 47D06, 93D15, 93C20

DOI. 10.1137/060671188

1. Introduction and background. It is well known that the flow dynamics in an irrigation river are generally described by nonlinear coupled hyperbolic partial differential equations and are called de Saint-Venant equations [37]. Nevertheless, by adopting several assumptions such as (i) neglecting lateral inflow and inertia terms, (ii) assuming small flow variations as well as small bed slope of the river, (iii) observing that the flow can be reasonably represented by a one-dimensional model, and (iv) observing that the range of flow values is somewhat reduced, one can consider the following system (see [3], [18], or [31] for more details):

$$\frac{\partial Q(x,t)}{\partial t} = \alpha \frac{\partial^2 Q(x,t)}{\partial x^2} - \beta \frac{\partial Q(x,t)}{\partial x} + w, \quad 0 < x < \ell,$$

where $Q$ is the water flow and $w$ is the lateral discharge, which is assumed to be constant for sake of simplicity ($w > 0$ represents the lateral inflow due to rains, for example, whereas $w < 0$ is the lateral outflow due to water withdrawals). The positive constants $\alpha, \beta, \ell$ and the variables $x$ and $t$ denote, respectively, the diffusion, the celerity, the length of the river, the distance in the downstream direction, and the time. This model is used in hydraulic engineering to describe dynamic behavior of the unsteady flow in a river for shallow water when the flow variations are not important (see [18] and the references therein).

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In practice, an irrigation system often consists of natural rivers to convey water released from an upstream dam to consumption locations, which are distributed along the reach. The system under consideration consists of a dam and one river reach with a measuring station at its downstream end. The control action variable is the upstream water flow, which means that the control acts so that the desired discharge is delivered. Therefore, we shall take the upstream flow as a control input variable, take the downstream flow as an output observation, and leave the downstream boundary condition as free (no control). In other words, the downstream influence is negligible for the mass transfer, that is to say, the flow variations are negligible, which is realistic in the case of long river reaches. This leads us to the following system:

\[
\begin{aligned}
\frac{\partial Q(x,t)}{\partial t} &= \alpha \frac{\partial^2 Q(x,t)}{\partial x^2} - \beta \frac{\partial Q(x,t)}{\partial x} + w, \quad 0 < x < \ell, \\
Q(0,t) &= u(t), \quad Q_x(\ell,t) = 0, \\
y(t) &= Q(\ell,t),
\end{aligned}
\]

where \(u(t)\) is the input (actuator) control and \(y(t)\) is the output (sensor) observation. Obviously, the actuator and sensor are not implemented at the same “location.” Hence the duality condition \(C = B^*\) does not arise, where \(B\) and \(C\) are, respectively, the input and output operators. This leads to a concrete example of a system with non-collocated actuators and sensors. As the reader may know, one of the main objectives in the management of irrigation systems is to keep the flow rate at the downstream end of the river close to a reference flow rate (target) fixed by the administration in charge of the river and calculated in order to ensure good environmental conditions for wildlife in the river.

Based on the above reasons, we shall investigate the stabilization and regulation problem of the system (1.1) with the following boundary proportional and integral controllers:

\[
\begin{aligned}
u(t) &= k_P y(t) + k_I \xi(t), \\
\xi(t) &= y(t) - y_r,
\end{aligned}
\]

where \(k_P, k_I \in \mathbb{R} \setminus \{0\}\) are, respectively, the proportional and the integral gains, whereas \(y_r\) is the constant reference signal to track. The role of the proportional gain \(k_P\) is to speed up the exponential decay rate for stability if necessary, while the role of integral gain \(k_I\) is to obtain the regulation property, namely, (i) reject disturbances \(w\), such as rains or withdrawals; (ii) make the output \(y(t) = Q(\ell,t)\) track a given constant reference signal \(y_r\) in spite of the constant perturbation \(w\); and (iii) achieve the exponential stability of the closed-loop system (1.1)–(1.2).

Note that the system (1.1) has been studied by many authors by means of approximation methods such as discretization (see, for instance, [18], [19], and [31] and the references therein), and hence the distributed character of (1.1) is not preserved. Recently, a qualitative analysis of (1.1) has been carried out in [2] by using the tools of infinite-dimensional systems theory [4]. In fact, stability and regulation results for the closed-loop system are proved when only an integral controller is applied in (1.2), that is, \(k_P = 0\). Note also that there is a considerable literature devoted to the design theory of integral and/or proportional controllers for infinite-dimensional systems. Indeed, in [34] and [41], the authors have been concerned with the existence of proportional and integral controllers for a class of infinite-dimensional systems with distributed controls. Moreover, the control operators are assumed to be bounded, and
hence their results cannot be applied to our problem. Furthermore, there exists a few research papers (see [35] and [36]) where the authors proposed an integral controller for a class of infinite-dimensional systems with boundary and distributed controls. However, in these works, the authors have been primarily interested in designing only integral controllers for systems with bounded boundary input and output operators via smooth (twice differentiable) controls. Once again, these restrictions do not permit us to use this approach. In a related area, there is a vast literature devoted to the frequency-domain robust controller design approach for systems described by transfer functions (see, for instance, [5], [11], [12], [21], [23], [27], [28] for low-gain control; [22] and [29] for high-gain control; and [24], [25], and [26] for regular systems subject to actuator nonlinearities). Unfortunately, we have met with some technical difficulties when using this approach. Finally, the reader may also find many articles where the general theoretical results obtained in the works cited above are applied or adapted to physical systems such as heat-exchangers and linearized Saint-Venant systems (see [6], [7], [8], [9], and [39]).

The main contribution of this paper is to adopt the Riesz basis approach and the shooting method in order to extend the results obtained in [2] on the system (1.1)–(1.2) without proportional gain \( k_P = 0 \) to the case when the feedback control (1.2) involves a proportional gain \( k_P \neq 0 \). Contrary to the work in [2], where \( k_P = 0 \) in (1.2), the advantage of the presence of the proportional gain \( k_P \) is to allow us to have more freedom in designing the stabilizing controller. Moreover, by a suitable choice of the proportional gain \( k_P \), one can improve, if necessary, the decay rate for the stability of the closed-loop system (1.1)–(1.2) compared to the case when only an integral controller is applied. It is important to note that it is not obvious to deduce neither the well-posedness nor stability results of the closed-loop system (1.1)–(1.2) from those obtained in [2]. This is due to the fact that the domain of the operator studied in [2] is perturbed, and hence the classical theory of perturbation of operators [13] fails. We also point out that the system (1.1)–(1.2) is nondissipative and has noncollocated actuator and sensor. Hence two major difficulties arise: first, how to show the \( C_0 \)-semigroup generation [33], and second, the stability for the system. To overcome this situation, the Riesz basis methodology [15], [16], [40] is used to show that the closed-loop system is a Riesz spectral system and generates an analytic semigroup. Concerning the exponential stability of the system (1.1)–(1.2), the shooting method is applied to assign the spectrum of the closed-loop system in the open left half-plane by means of an appropriate choice of the proportional and integral gains \( k_P \) and \( k_I \). We should note that the existence results of such gains are theoretic, whereas their tuning is not dealt with here and still remain an open problem. Furthermore, the output regulation is guaranteed independently of any constant (known or unknown) perturbation.

The paper is organized as follows. In the next section, some preliminary results are stated for the uncontrolled system. We also convert the closed-loop system (1.1)–(1.2) into an evolution equation in an appropriate Hilbert space and then state the main results of this article. Sections 3 and 4 are devoted to the proof of the main results. First, we deal with the eigenvalue problem, and asymptotic expansions of both eigenvalues and eigenfunctions of the system are explicitly presented. Next, we use the Green’s function approach to obtain an estimate of the resolvent which leads to the completeness of the root subspace. Then, the Riesz basis property and the \( C_0 \)-semigroup generation of the system are proved. Finally, we establish, under certain conditions on the feedback gains \( k_P \) and \( k_I \), the exponential stability and deduce the regulation of the closed-loop system (1.1)–(1.2). These results are illustrated by some
2. Preliminaries and main results. First, let us consider the uncontrolled system \((u(t) = 0)\) with no disturbances:

\[
\begin{cases}
\frac{\partial Q(x, t)}{\partial t} = \alpha \frac{\partial^2 Q(x, t)}{\partial x^2} - \beta \frac{\partial Q(x, t)}{\partial x}, & 0 < x < \ell, \\
Q(0, t) = Q_x(\ell, t) = 0.
\end{cases}
\]

Taking the Hilbert state space \(H_0 = L^2(0, \ell)\) equipped with the usual inner product, the system (2.1) can be written in the following abstract form:

\[
\dot{Q}(t) = A_0 Q(t),
\]

where \(A_0\) is an unbounded linear operator defined by

\[
D(A_0) := \left\{ Q \in H^2(0, \ell); Q(0) = Q_x(\ell) = 0 \right\} \quad \text{and} \quad A_0 := \alpha \frac{\partial^2}{\partial x^2} - \beta \frac{\partial}{\partial x}.
\]

Clearly, \(\lambda^0\) is an eigenvalue of \(A_0\) if and only if the system

\[
(2.3) \quad \alpha f_0'' - \beta f_0' - \lambda^0 f_0 = 0,
\]

\[
(2.4) \quad f_0(0) = f_0'(\ell) = 0
\]

has a nonzero solution.

The principal properties of the operator \(A_0\) are summarized as follows (see [2] for details).

**Lemma 2.1.** The operator \(A_0\) generates an exponentially stable \(C_0\)-semigroup of contractions \(S_0(t)\) on \(H_0\). Moreover, the spectrum \(\sigma(A_0)\) of \(A_0\) consists of negative real numbers of the form \(-\frac{\alpha \tau^2}{\ell^2} - \frac{\beta^2}{4\alpha}\), where \(\tau\) is a nonzero solution to

\[
(2.5) \quad \beta \ell \sin \tau + 2\alpha \tau \cos \tau = 0.
\]

Consider now the function \(x \mapsto f_0(x, \lambda)\) as the solution of (2.3) subject to the conditions

\[
(2.6) \quad f_0'(\ell, \lambda) = 0, \quad f_0(\ell, \lambda) = 1.
\]

Then, it is obvious that the zeros of \(f_0(0, \lambda)\) are the eigenvalues of \(A_0\), which are real and negative by Lemma 2.5, namely,

\[
\cdots < \lambda^0_n < \cdots < \lambda^0_1 < 0.
\]

We have the following result, whose proof is given in [2].

**Proposition 2.2.** Consider the normalized solution of (2.3)–(2.4) by \(f_0(\ell, \lambda) = 1\). Then the zeros of \(f_0(0, \lambda)\) are the eigenvalues \(\lambda^0_i\) of the operator \(A_0\), which are negative real numbers and simple. In addition, we have

\[
(2.7) \quad f_0(0, \lambda) = \prod_{i=1}^{\infty} \left(1 - \frac{\lambda}{\lambda^0_i}\right).
\]

Denote

\[
B(\lambda) := \lambda f_0(0, \lambda) = \lambda \prod_{i=1}^{\infty} \left(1 - \frac{\lambda}{\lambda^0_i}\right).
\]
The infinite product $B(\lambda)$ will play a crucial role in the stability of the closed-loop system (1.1)–(1.2). It follows from Lemma 2.5 and Proposition 2.2 that the zeros of $B$, namely, $\lambda^0_i$, are simple and negative. Furthermore, between two consecutive zeros, the function $B$ attains its local maxima and minima. Thus, let

$$M_k = \min_{\lambda^0_{2k+1} < \lambda < \lambda^0_{2k}} B(\lambda)$$

denote negative minima when $B$ is negative between two zeros, and let $m_k$ be the ordered set of values $M_k$ as follows (see Figure 1):

$$m_2 < m_1 < 0.$$  

Using the same arguments as in [2], one can prove the following lemma.

**Lemma 2.3.** The polynomials $B_n(\lambda) := \lambda \prod_{i=1}^n (1 - \frac{\lambda}{\lambda^0_i})$ converge uniformly to $B(\lambda)$ in any compact domain of the complex plane. Furthermore, any compact domain of the complex plane, not containing the zeros of $B(\cdot) - ((\cdot)k_p + k_I)$ on its boundary, contains the same number of zeros of $B(\cdot) - ((\cdot)k_p + k_I)$ and $B(\cdot) - ((\cdot)k_p + k_I)$ for $n$ large enough.

Now, let us convert the closed-loop system (1.1)–(1.2) into an evolution equation in an appropriate Hilbert space and then state the basic properties of the system operator. Clearly, the closed-loop system (1.1)–(1.2) is governed in the “augmented” state space

$$\mathcal{H} := L^2(0, \ell) \times \mathbb{C},$$

equipped with the inner product induced by the norm

$$\| (f, \xi) \|^2 = \int_0^\ell |f(x)|^2dx + |\xi|^2 \quad \forall (f, \xi) \in \mathcal{H},$$
by the system
\begin{equation}
\dot{\phi}(t) = A_{PI} \phi(t) + (w, -y_r).
\end{equation}

Here $\phi = (f, \xi)$ and $A_{PI}$ is an unbounded linear operator defined by
\begin{equation}
D(A_{PI}) := \left\{ (f, \xi) \in H^2(0, \ell) \times \mathbb{C}; \; f(0) = k_P f(\ell) + k_I \xi; \; f'(\ell) = 0 \right\}
\end{equation}
and
\begin{equation}
A_{PI} (f, \xi) := \left( \alpha f'' - \beta f', f(\ell) \right) \text{ for any } (f, \xi) \in D(A_{PI}).
\end{equation}

Recall that $\alpha$, $\beta$, and $\ell$ are positive constants and the feedback gains $k_P$ and $k_I$ are nonzero numbers.

Now, one can show that, given $(g, \eta) \in \mathcal{H}$, the equation $A_{PI}(f, \xi) = (g, \eta)$ has a unique solution $(f, \xi) \in D(A_{PI})$ given by
\begin{equation}
\begin{cases}
    f(x) = \eta + \frac{1}{\beta} \int_x^\ell \left( 1 - e^{\omega(x-\xi)} \right) g(\xi) \, d\xi, \\
    \xi = \frac{1}{k_I} (f(0) - k_P f(\ell)),
\end{cases}
\end{equation}
where $\omega = \beta/\alpha$. Therefore, the operator $(A_{PI})^{-1} \in \mathcal{L}(\mathcal{H})$. Moreover, by the Sobolev embedding theorem [1], we deduce that $(A_{PI})^{-1}$ is compact on $\mathcal{H}$, and hence the spectrum $\sigma(A_{PI})$ consists of isolated eigenvalues only [13].

Before stating our main results, let us recall that a nonzero $Y$, of a Hilbert space $H$, is called a generalized eigenvector of a linear operator $A$, corresponding to an eigenvalue $\lambda$ (with finite algebraic multiplicity) of $A$, if there is a positive integer $n$ such that $(\lambda - A)^n Y = 0$.

Let $\text{Sp}(A)$ be the root subspace of a linear operator $A$ which is defined as the closed subspace spanned by all generalized eigenvectors of $A$. A sequence in $H$ is said to be complete if its linear span is dense in $H$. Also, a sequence in $H$ is called minimal if each element of this sequence lies outside the closed linear span of the remaining elements. In turn, two sequences $\{e_i\}$ and $\{e_i^*\}$ are said to be biorthogonal in $H$ if
\begin{equation}
\langle e_i, e_j^* \rangle = \delta_{ij} = \begin{cases} 
    1, & i = j, \\
    0, & i \neq j,
\end{cases}
\end{equation}
for every $i$ and $j$. It is well known that for a given sequence $\{e_i\}$, a biorthogonal sequence $\{e_i^*\}$ exists if and only if $\{e_i\}$ is minimal, and $\{e_i^*\}$ is uniquely determined if and only if $\{e_i\}$ is complete.

Now, a sequence $\{e_i\}_{i=1}^\infty$ is called a Bessel sequence in $H$ if for any $x \in H$, the series $\{\langle x, e_i \rangle\}_{i=1}^\infty \in \ell^2$. On the other hand, a sequence $\{e_i\}_{i=1}^\infty$ is called a basis for $H$ if any element $x \in H$ has a unique representation,
\begin{equation}
x = \sum_{i=1}^\infty a_i e_i,
\end{equation}
and the convergence of the series is in the norm of $H$. Finally, a sequence $\{e_i\}_{i=1}^\infty$ with a biorthogonal sequence $\{e_i^*\}_{i=1}^\infty$ is called a Riesz basis for $H$ if $\{e_i\}_{i=1}^\infty$ is an
It is also well known that which are complex conjugate numbers with positive or negative real parts. 

The results of the eigenvalue problem related to the operator system. For sake of clarity, we divide this section into three parts. 

The theorems 2.4 and 2.5, namely, the well-posedness and the Riesz basis generation of 

Now, we are able to state the main results of this work. Indeed, the first main result is related to the Riesz basis property. 

THEOREM 2.4. Let |k_p| ≠ 1, where k_p := k_p e^{\frac{1}{a} t}. Then, the generalized eigenfunctions of the operator \( A_{PI} \) form a Riesz basis in \( \mathcal{H} \). In turn, if |k_p| = 1, then the generalized eigenfunctions of \( A_{PI} \) form, in \( \mathcal{H} \), a Riesz basis with parentheses. 

The second result is mainly concerned with the properties of the semigroup generated by the operator \( A_{PI} \). 

THEOREM 2.5. The operator \( A_{PI} \) defined by (2.10)–(2.11) generates a \( C_0 \)-semigroup \( S_{PI}(t) \) in \( \mathcal{H} \). Therefore, \( S_{PI}(t) \) satisfies the spectrum-determined growth condition \( S(\sigma_{A_{PI}}) = \omega(\sigma_{A_{PI}}) \), where \( S(\sigma_{A_{PI}}) = \sup_{\lambda \in \sigma_{A_{PI}}} \text{Re} \lambda \) is the spectral bound and \( \omega(\sigma_{A_{PI}}) \) is the growth order of the semigroup \( S_{PI}(t) \). Moreover, \( S_{PI}(t) \) is an analytic semigroup in \( \mathcal{H} \). 

Finally, regarding the third main result, we use notation from (2.8) and Figure 1 to state the stability and regulation properties of the closed-loop system. 

THEOREM 2.6. (i) If \( m_1 < k_1 < 0 \) and \( k_p \) is a sufficiently small positive number, then the spectrum \( \sigma(\sigma_{A_{PI}}) \) consists of negative real numbers. 

(ii) If \( m_2 < k_2 < m_1 < 0 \) and \( k_p \) is a positive number sufficiently small, then the spectrum \( \sigma(\sigma_{A_{PI}}) \) of the operator \( A_{PI} \) consists of negative real numbers, except two which are complex conjugate numbers with positive or negative real parts. 

Therefore, in both cases where the spectrum of the operator \( A_{PI} \), defined by (2.10)–(2.11) has negative real part, the analytic semigroup \( S_{PI}(t) \) is exponentially stable. Moreover, for any initial data \( \varphi_0 = (Q_0, \xi_0) \in \mathcal{D}(A_{PI}) \) and for any constant perturbation \( w \) in \( \mathcal{H} \), we have the output regulation 

\[
\lim_{t \to \infty} y(t) = \lim_{t \to \infty} Q(\ell, t) = y_r.
\]

Furthermore, the closed-loop system (1.1)–(1.2) is exponentially stable in \( \mathcal{H} \) in spite of the constant perturbation \( w \) in \( \mathcal{H} \). 

3. Proof of Theorems 2.4 and 2.5. In this section, we are going to show Theorems 2.4 and 2.5, namely, the well-posedness and the Riesz basis generation of the system. For sake of clarity, we divide this section into three parts. 

3.1. Eigenvalue problem. We shall establish, in this subsection, the basic results of the eigenvalue problem related to the operator \( A_{PI} \). From the eigenvalue equation \( A_{PI}(f, \xi) = \lambda(f, \xi) \), we have the characteristic equation in \( \lambda \): 

\[
\begin{align*}
\lambda''(x) - \frac{2}{a} \lambda'(x) - \frac{1}{a} \lambda f(x) &= 0, & 0 < x < \ell, \\
(\lambda k_p + k_f)\ell f(\ell) &= \lambda f(0), & f'(\ell) = 0.
\end{align*}
\]

In order to solve the above equation, we introduce the following transformation that can translate the interval \([0, \ell]\) into \([0, 1]\): 

\[
x = z\ell, \quad f(x) = g(z), \quad z \in [0, 1].
\]
Then, (3.1) changes into

\[
\begin{align*}
\{ g''(z) - \frac{\ell^2}{\alpha^2} g'(z) - \frac{\ell^2}{\alpha^2} \lambda g(z) &= 0, & 0 < z < 1, \\
(\lambda k_P + k_I)g(1) &= \lambda g(0), & g'(1) &= 0.
\end{align*}
\]

Let

\[
\tau_1(\lambda) = \frac{\ell \beta + \ell \sqrt{\beta^2 + 4 \alpha \lambda}}{2\alpha}, \quad \tau_2(\lambda) = \frac{\ell \beta - \ell \sqrt{\beta^2 + 4 \alpha \lambda}}{2\alpha}.
\]

Clearly, \( e^{\tau_1 z} \) and \( e^{\tau_2 z} \) are two independent solutions of \( g''(z) - \frac{\ell^2}{\alpha^2} g'(z) - \frac{\ell^2}{\alpha^2} \lambda g(z) = 0, \)
and thus the general solution form of (3.3) can be given by

\[
g(z) = c_1 e^{\tau_1 z} + c_2 e^{\tau_2 z},
\]

where \( c_1 \) and \( c_2 \) satisfy the following system:

\[
\begin{align*}
(\lambda k_P + k_I)(c_1 e^{\tau_1} + c_2 e^{\tau_2}) &= \lambda (c_1 + c_2), \\
c_1 \tau_1 e^{\tau_1} + c_2 \tau_2 e^{\tau_2} &= 0.
\end{align*}
\]

Hence, (3.3) has a nontrivial solution if and only if

\[
\det(\Delta(\lambda)) = 0,
\]

where \( \Delta(\lambda) \) is the coefficient matrix given by

\[
\Delta(\lambda) := \begin{bmatrix} (\lambda k_P + k_I) e^{\tau_1} - \lambda (\lambda k_P + k_I) e^{\tau_2} - \lambda & \tau_1 e^{\tau_1} \\ \tau_1 e^{\tau_1} & \tau_2 e^{\tau_2} \end{bmatrix}.
\]

By a direct computation, we have

\[
\det(\Delta(\lambda)) = (\tau_2 - \tau_1)(\lambda k_P + k_I) e^{\tau_1 + \tau_2} - \lambda (\tau_2 e^{\tau_2} - \tau_1 e^{\tau_1})
\]

\[
= -\frac{\ell \sqrt{\beta^2 + 4 \alpha \lambda}}{\alpha} (\lambda k_P + k_I) e^{(\beta)^2} - \lambda (\tau_2 e^{\tau_2} - \tau_1 e^{\tau_1}).
\]

Let \( \lambda := \frac{\alpha \rho^2 - \beta^2}{2\alpha \ell^2} \), with \( \rho \in \mathbb{C} \) and \( \arg \rho \in [-\pi/2, \pi/2) \). Then \( \tau_1(\lambda) \) and \( \tau_2(\lambda) \), defined by (3.4), become

\[
\tau_1(\rho) = \frac{\ell \beta}{2\alpha} + \rho, \quad \tau_2(\rho) = \frac{\ell \beta}{2\alpha} - \rho.
\]

We also have

\[
e^{-\frac{\beta}{\ell^2}} \det(\Delta(\rho)) = -2\rho \left( \frac{\alpha \rho^2}{\ell^2} - \frac{\beta^2}{4\alpha} \right) \tilde{k}_P - 2\rho \tilde{k}_I
\]

\[
- \left( \frac{\alpha \rho^2}{\ell^2} - \frac{\beta^2}{4\alpha} \right) \left( \frac{\ell \beta}{2\alpha} e^{-\rho} - \rho e^{-\rho} - \frac{\ell \beta}{2\alpha} e^\rho - \rho e^\rho \right)
\]

\[
= \frac{\alpha \rho^3}{\ell^2} (e^{-\rho} + e^\rho - 2\tilde{k}_P) - \frac{\beta \rho^2}{2\ell} (e^{-\rho} - e^\rho)
\]

\[
- \frac{\beta^2}{4\alpha} \rho \left( e^{-\rho} + e^\rho - 2\tilde{k}_P + \frac{8\alpha}{\beta^2} \tilde{k}_I \right) + \frac{\ell \beta^3}{8\alpha^2} (e^{-\rho} - e^\rho),
\]
where

\begin{equation}
\tilde{k}_P := k_P e^{\frac{1}{2} \alpha}, \quad \tilde{k}_I := k_I e^{\frac{1}{2} \alpha}.
\end{equation}

Let us summarize the previous results in the following lemma.

**Lemma 3.1.** Let \( \lambda := \alpha \rho^2 - \frac{\beta^2}{4\rho^2} \), with \( \arg \rho \in [-\pi/2, \pi/2) \). Then the characteristic determinant \( \det(\Delta(\rho)) \) has the following form:

\begin{equation}
e^{-\frac{\alpha \rho^3}{\ell^2}} \det(\Delta(\rho)) = \frac{\alpha \rho^3}{\ell^2} \left( e^{-\rho} + e^{\rho} - 2\tilde{k}_P \right) - \frac{\beta^2}{4\alpha} \rho \left( e^{-\rho} + e^{\rho} - 2\tilde{k}_P + \frac{8\alpha}{\beta^2} \tilde{k}_I \right) + \frac{\ell \beta^3}{8\alpha^2} (e^{-\rho} - e^\rho),
\end{equation}

with \( \tilde{k}_P \) and \( \tilde{k}_I \) being given by (3.8).

Then, we have the following.

**Theorem 3.2.** Let \( |\tilde{k}_P| > 1 \). Then the eigenvalues \( \lambda_n \) of the operator \( A_{PI} \) are the solutions of the eigenvalue problem (3.3) and have the following asymptotic expansion: for \( s = 1, 2 \),

\begin{equation}
\lambda_{ns} = -\left( \beta + \frac{\beta^2}{4\alpha} \right) + \frac{(\ln |\tilde{k}_{Ps}|)^2 - (2n\pi + \delta)^2 + i(4n\pi + 2\delta) \ln |\tilde{k}_{Ps}|}{\alpha^{-1}\ell^2} + \mathcal{O}(n^{-1}),
\end{equation}

where \( n \) are positive integers,

\begin{equation}
\tilde{k}_{P1} := \tilde{k}_P + \sqrt{\tilde{k}_P^2 - 1}, \quad \tilde{k}_{P2} := \tilde{k}_P - \sqrt{\tilde{k}_P^2 - 1},
\end{equation}

and

\begin{equation}
\delta := \begin{cases} 0 & \text{when } \tilde{k}_P \geq 1; \\
\pi & \text{when } \tilde{k}_P \leq -1.
\end{cases}
\end{equation}

In turn, if \( |\tilde{k}_P| = 1 \), then the eigenvalues \( \lambda_{n1} \) and \( \lambda_{n2} \) are not separable when their moduli are large enough and have the following asymptotic expression:

\begin{equation}
\lambda_{ns} = -\frac{\beta^2}{4\alpha} - \frac{\alpha (2n\pi + \delta)^2}{\ell^2} + \mathcal{O}(n^{-1}), \quad s = 1, 2.
\end{equation}

**Proof.** Using (3.9), the characteristic equation \( \det(\Delta(\rho)) = 0 \) means that \( \rho \), with \( \arg \rho \in [-\pi/2, \pi/2) \), satisfies the following:

\begin{equation}
e^\rho + e^{-\rho} - 2\tilde{k}_P - \frac{1}{2} \frac{\ell \beta}{\alpha} (e^{-\rho} - e^\rho) \rho^{-1} + \mathcal{O}(\rho^{-2}) = 0,
\end{equation}

which leads to

\begin{equation}
e^\rho + e^{-\rho} - 2\tilde{k}_P + \mathcal{O}(\rho^{-1}) = 0.
\end{equation}

Assume now that \( |\tilde{k}_P| > 1 \). Then, it follows that the solutions of the equation

\begin{equation}
e^\rho + e^{-\rho} - 2\tilde{k}_P = 0
\end{equation}

are given by

\begin{equation}
\tilde{\rho}_{ns} = \ln |\tilde{k}_{Ps}| + 2n\pi i + \delta i, \quad s = 1, 2, \quad n \in \mathbb{N},
\end{equation}

where

\begin{equation}
\tilde{k}_P := k_P e^{\frac{1}{2} \alpha}, \quad \tilde{k}_I := k_I e^{\frac{1}{2} \alpha}.
\end{equation}
where $\delta$ is as defined in (3.12). Rouché’s theorem can be applied to (3.15) to obtain

\begin{equation}
\rho_{ns} = \tilde{\rho}_{ns} + \alpha_{ns}, \quad \alpha_{ns} = O(n^{-1}), \quad s = 1, 2,
\end{equation}

for sufficiently large positive integers $n$. Substituting $\rho_{ns}$ into (3.14), we get

\[ e^{\tilde{\rho}_{ns} + \alpha_{ns}} + e^{-\tilde{\rho}_{ns} - \alpha_{ns}} - 2\tilde{k}_P - \frac{1}{2} \beta \left( e^{-\tilde{\rho}_{ns} - \alpha_{ns}} - e^{\tilde{\rho}_{ns} + \alpha_{ns}} \right) \tilde{\rho}_{ns}^{-1} + O(\rho_{ns}^{-2}) = 0. \]

Note that

\[ e^{\tilde{\rho}_{ns}} + e^{-\tilde{\rho}_{ns}} - 2\tilde{k}_P = 0, \quad e^{\tilde{\rho}_{ns}} = \tilde{k}_P s, \]

and hence

\[ e^{\tilde{\rho}_{ns} + \alpha_{ns}} + e^{-\tilde{\rho}_{ns} - \alpha_{ns}} = e^{\alpha_{ns}} \left[ 2\tilde{k}_P - e^{-\tilde{\rho}_{ns}} \right] + e^{-\tilde{\rho}_{ns} - \alpha_{ns}} \]

\[ = 2\tilde{k}_P e^{\alpha_{ns}} - e^{\alpha_{ns}} e^{-\tilde{\rho}_{ns}} + e^{-\tilde{\rho}_{ns} - \alpha_{ns}} \]

\[ = 2\tilde{k}_P \left[ 1 + \alpha_{ns} + O(n^{-2}) \right] - e^{-\tilde{\rho}_{ns}} \left[ e^{\alpha_{ns}} - e^{-\alpha_{ns}} \right] \]

\[ = 2\tilde{k}_P + 2\alpha_{ns}\tilde{k}_P + O(n^{-2}) - e^{-\tilde{\rho}_{ns}} \left[ 2\alpha_{ns} + O(n^{-2}) \right] \]

\[ = 2\tilde{k}_P + 2\alpha_{ns}\tilde{k}_P + O(n^{-2}) - \tilde{k}_P^{-1} \left[ 2\alpha_{ns} + O(n^{-2}) \right] \]

\[ = 2\tilde{k}_P + 2\alpha_{ns} \left( \tilde{k}_P - \tilde{k}_P^{-1} \right) + O(n^{-2}), \]

where we have also expanded the exponential functions according to their Taylor series. Similarly,

\[ e^{\tilde{\rho}_{ns} + \alpha_{ns}} - e^{-\tilde{\rho}_{ns} - \alpha_{ns}} = e^{\alpha_{ns}} \left[ 2\tilde{k}_P - e^{-\tilde{\rho}_{ns}} \right] - e^{-\tilde{\rho}_{ns} - \alpha_{ns}} \]

\[ = 2k_P e^{\alpha_{ns}} - e^{\alpha_{ns}} e^{-\tilde{\rho}_{ns}} - e^{-\tilde{\rho}_{ns} - \alpha_{ns}} \]

\[ = 2k_P \left[ 1 + \alpha_{ns} + O(n^{-2}) \right] - e^{-\tilde{\rho}_{ns}} \left[ e^{\alpha_{ns}} + e^{-\alpha_{ns}} \right] \]

\[ = 2k_P + 2\alpha_{ns}k_P + O(n^{-2}) - e^{-\tilde{\rho}_{ns}} \left[ 2 + O(n^{-2}) \right] \]

\[ = 2k_P + 2\alpha_{ns}k_P + O(n^{-2}) - k_P^{-1} \left[ 2 + O(n^{-2}) \right] \]

\[ = 2 \left( k_P - k_P^{-1} \right) + 2k_P \alpha_{ns} + O(n^{-2}). \]

These two estimates lead us to write

\[ 0 = e^{\tilde{\rho}_{ns} + \alpha_{ns}} + e^{-\tilde{\rho}_{ns} - \alpha_{ns}} - 2\tilde{k}_P - \frac{1}{2} \beta \left( e^{-\tilde{\rho}_{ns} - \alpha_{ns}} - e^{\tilde{\rho}_{ns} + \alpha_{ns}} \right) \tilde{\rho}_{ns}^{-1} + O(\rho_{ns}^{-2}) \]

\[ = 2k_P + 2\alpha_{ns} \left( \tilde{k}_P - k_P^{-1} \right) - 2k_P \]

\[ + \frac{1}{2} \beta \left( 2k_P - k_P^{-1} \right) \left( k_P - k_P^{-1} \right) + 2k_P \alpha_{ns} + O(n^{-2}) \tilde{\rho}_{ns}^{-1} + O(n^{-2}) \]

\[ = 2\alpha_{ns} \left( \tilde{k}_P - k_P^{-1} \right) + \frac{1}{2} \beta \left( 2k_P - k_P^{-1} \right) \left( k_P - k_P^{-1} \right) \tilde{\rho}_{ns}^{-1} + O(n^{-2}) \]

\[ = 2\alpha_{ns} \left( \tilde{k}_P - k_P^{-1} \right) + \frac{1}{2} \beta \left( k_P - k_P^{-1} \right) \tilde{\rho}_{ns}^{-1} + O(n^{-2}). \]
Thus, we obtain
\[ \alpha_{ns} = -\frac{\ell \beta}{2a\rho_{ns}} + O(n^{-2}) = -\frac{\ell \beta}{2a \left( \ln |\tilde{k}_{Ps}| + 2n\pi i + \delta i \right)} + O(n^{-2}). \]

This, together with (3.16)–(3.17), yields
\[ (3.18) \quad \rho_{ns} = \ln |\tilde{k}_{Ps}| + 2n\pi i + \delta i - \frac{\ell \beta}{2a \left( \ln |\tilde{k}_{Ps}| + 2n\pi i + \delta i \right)} + O(n^{-2}), \]

and so
\[ \rho_{ns}^2 = (\ln |\tilde{k}_{Ps}|)^2 - (2n\pi + \delta)^2 + i(4n\pi + 2\delta) \ln |\tilde{k}_{Ps}| - \frac{\ell \beta}{\alpha} + O(n^{-1}). \]

Now, using the fact that \( \lambda_{ns} = \frac{\alpha \rho_{ns}^2}{2} - \frac{\beta^2}{4\pi} \), the desired result (3.10) is directly obtained.

Finally, when \( |\tilde{k}_{P}| = 1 \), similar arguments permit us to claim that \( \lambda_{n1} \) and \( \lambda_{n2} \), for \( n \in \mathbb{N} \), are not separable when their moduli are large enough and have the asymptotic expression given by (3.13). We omit the details here.

The following result deals with the case \( |\tilde{k}_{P}| < 1 \).

**Theorem 3.3.** Let \( |\tilde{k}_{P}| < 1 \). Then the eigenvalues \( \lambda_n \) of the operator \( A_{Pl} \) are the solutions of the eigenvalue problem (3.3) and have the following asymptotic expansion: for \( s = 1, 2 \),
\[ (3.19) \quad \lambda_{ns} = -\left( \frac{\beta}{\ell} + \frac{\beta^2}{4\alpha} \right) - \frac{\alpha(2n\pi + \theta_s)^2}{\ell^2} + O(n^{-1}), \]

where \( n \) are positive integers and
\[ (3.20) \quad \theta_1 := \tan^{-1} \left( \frac{\sqrt{1 - \tilde{k}_{P}^2}}{\tilde{k}_{P}} \right), \quad \theta_2 := -\theta_1. \]

**Proof.** Let \( \rho \in \mathbb{C} \) with \( \arg \rho \in [-\pi/2, \pi/2) \). When \( |\tilde{k}_{P}| < 1 \), the equation
\[ e^{\rho} + e^{-\rho} - 2\tilde{k}_{P} = 0 \]

has solutions
\[ (3.21) \quad \tilde{\rho}_{ns} = \theta_s i + 2n\pi i, \quad s = 1, 2, \quad n \in \mathbb{N}. \]

Applying Rouché’s theorem to (3.15) yields
\[ (3.22) \quad \rho_{ns} = \tilde{\rho}_{ns} + \alpha_{ns}, \quad \alpha_{ns} = O(n^{-1}), \quad s = 1, 2, \]

for sufficiently large positive integers \( n \). Next, inserting \( \rho_{ns} \) into (3.14), we obtain, after a careful computation,
\[ e^{\tilde{\rho}_{ns} + \alpha_{ns}} + e^{-\tilde{\rho}_{ns} - \alpha_{ns}} - 2\tilde{k}_{P} - \frac{1}{2} \frac{\ell \beta}{\alpha} \left( e^{-\tilde{\rho}_{ns} - \alpha_{ns}} - e^{\tilde{\rho}_{ns} + \alpha_{ns}} \right) \rho_{ns}^{-1} + O(\rho_{ns}^{-2}) = 0. \]

Then, since
\[ e^{\tilde{\rho}_{ns}} + e^{-\tilde{\rho}_{ns}} - 2\tilde{k}_{P} = 0, \quad e^{\tilde{\rho}_{ns}} = e^{i\theta_s} = \tilde{k}_{P} + i\sqrt{1 - \tilde{k}_{P}^2}, \]

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one can get, as for the case $|\tilde{k}_p| > 1$, the estimates
\[
e^{\tilde{\rho}_{ns} + \alpha_{ns}} + e^{-\tilde{\rho}_{ns} - \alpha_{ns}} = e^{\alpha_{ns}} \left[ 2\tilde{k}_p - e^{-\tilde{\rho}_{ns}} \right] + e^{-\tilde{\rho}_{ns} - \alpha_{ns}}
\]
\[
= 2\tilde{k}_p e^{\alpha_{ns}} - e^{\alpha_{ns}} e^{-\tilde{\rho}_{ns}} + e^{-\tilde{\rho}_{ns} - \alpha_{ns}}
\]
\[
= 2\tilde{k}_p \left[ 1 + \alpha_{ns} + O(n^{-2}) \right] - e^{-\tilde{\rho}_{ns}} \left[ e^{\alpha_{ns}} - e^{-\alpha_{ns}} \right]
\]
\[
= 2\tilde{k}_p + 2\alpha_{ns} \tilde{k}_p + O(n^{-2}) - e^{-\tilde{\rho}_{ns}} \left[ 2\alpha_{ns} + O(n^{-2}) \right]
\]
\[
= 2\tilde{k}_p + 2\alpha_{ns} \tilde{k}_p + O(n^{-2}) - \left( \tilde{k}_p - i\sqrt{1 - \tilde{k}_p^2} \right) \left[ 2\alpha_{ns} + O(n^{-2}) \right]
\]
\[
= 2\tilde{k}_p + 2i\alpha_{ns} \sqrt{1 - \tilde{k}_p^2} + O(n^{-2})
\]
and
\[
e^{-\tilde{\rho}_{ns} + \alpha_{ns}} - e^{-\tilde{\rho}_{ns} - \alpha_{ns}} = e^{\alpha_{ns}} \left[ 2\tilde{k}_p - e^{-\tilde{\rho}_{ns}} \right] - e^{-\tilde{\rho}_{ns} - \alpha_{ns}}
\]
\[
= 2\tilde{k}_p e^{\alpha_{ns}} - e^{\alpha_{ns}} e^{-\tilde{\rho}_{ns}} - e^{-\tilde{\rho}_{ns} - \alpha_{ns}}
\]
\[
= 2\tilde{k}_p \left[ 1 + \alpha_{ns} + O(n^{-2}) \right] - e^{-\tilde{\rho}_{ns}} \left[ e^{\alpha_{ns}} + e^{-\alpha_{ns}} \right]
\]
\[
= 2\tilde{k}_p + 2\alpha_{ns} \tilde{k}_p + O(n^{-2}) - e^{-\tilde{\rho}_{ns}} \left[ 2 + O(n^{-2}) \right]
\]
\[
= 2\tilde{k}_p + 2\alpha_{ns} \tilde{k}_p + O(n^{-2}) - \left( \tilde{k}_p - i\sqrt{1 - \tilde{k}_p^2} \right) \left[ 2 + O(n^{-2}) \right]
\]
\[
= 2i\sqrt{1 - \tilde{k}_p^2} + 2\tilde{k}_p \alpha_{ns} + O(n^{-2}).
\]
Hence
\[
0 = e^{\tilde{\rho}_{ns} + \alpha_{ns}} + e^{-\tilde{\rho}_{ns} - \alpha_{ns}} - 2\tilde{k}_p - \frac{\ell \beta}{2 \alpha} \left( e^{-\tilde{\rho}_{ns} - \alpha_{ns}} - e^{\tilde{\rho}_{ns} + \alpha_{ns}} \right) \tilde{\rho}_{ns}^{-1} + O(\rho_{ns}^{-2})
\]
\[
= 2\tilde{k}_p + 2i\alpha_{ns} \sqrt{1 - \tilde{k}_p^2} - 2\tilde{k}_p - \frac{\ell \beta}{2 \alpha} \left( 2i\sqrt{1 - \tilde{k}_p^2} + 2\tilde{k}_p \alpha_{ns} + O(n^{-2}) \right) \tilde{\rho}_{ns}^{-1} + O(n^{-2})
\]
\[
= 2i\alpha_{ns} \sqrt{1 - \tilde{k}_p^2} + \frac{\ell \beta}{2 \alpha} \left( 2i\sqrt{1 - \tilde{k}_p^2} + 2\tilde{k}_p \alpha_{ns} \right) \tilde{\rho}_{ns}^{-1} + O(n^{-2})
\]
\[
= 2i\alpha_{ns} \sqrt{1 - \tilde{k}_p^2} + \frac{\ell \beta}{\alpha} \left( 1 - \tilde{k}_p^2 \tilde{\rho}_{ns}^{-1} + O(n^{-2}) \right),
\]
which, together with (3.21), gives
\[
\alpha_{ns} = -\frac{\ell \beta}{2\alpha} \tilde{\rho}_{ns}^{-1} + O(n^{-2}) = -\frac{\ell \beta}{2\alpha (\theta_{ns} + 2n\pi i)} + O(n^{-2}).
\]
Therefore
\[
\rho_{ns} = \theta_{ns} + 2n\pi i - \frac{\ell \beta}{2\alpha (\theta_{ns} + 2n\pi i)} + O(n^{-2}),
\]
\[ \rho_{ns}^2 = - (2n\pi + \theta_s)^2 - \frac{\kappa \beta}{\alpha} + O(n^{-1}). \]

Finally, since \( \lambda_{ns} = \frac{\alpha \rho_{ns}^2}{\ell^2} - \frac{\beta^2}{4n^2} \), one can deduce (3.19). This completes the proof. \( \square \)

We are now in a position to investigate the asymptotic behavior of the eigenfunctions. The result is described as follows.

**Theorem 3.4.** Let \( \sigma(\mathcal{A}_p) = \{\lambda_{n1}, \lambda_{n2}, n \in \mathbb{N}\} \) be the eigenvalues of \( \mathcal{A}_p \). If \( |\tilde{k}_p| > 1 \) (respectively, \( |\tilde{k}_p| < 1 \)), then \( \lambda_n = \frac{\alpha \rho_{ns}^2}{\ell^2} - \frac{\beta^2}{4n^2} \), with \( \rho_n \in \mathbb{C} \) and \( \arg \rho_n \in [-\pi/2, \pi/2] \), are given by (3.10) and (3.18) (respectively, (3.19) and (3.23)). Furthermore, the corresponding eigenfunctions \( \{f_{n1}(\xi_1), (f_{n2}, \xi_2)\} \) have the following asymptotics: for \( s = 1, 2 \),

\[ f_{ns}(x) = g_{ns}(z) = e^{\frac{\ell}{2n}z} e^{-\rho_{ns}(1-z)} + e^{\frac{\ell}{2n}z} e^{\rho_{ns}(1-z)} + O(n^{-1}) \]

\[ \xi_{ns} = \frac{g_{ns}(0) - k_p g_{ns}(1)}{k_I} = O(n^{-1}) \]

for sufficiently large positive integer \( n \), where \( f_{ns}(x) = g_{ns}(z) \) with \( x = z\ell \) given in (3.2). Moreover, \( \{f_{n1}(\xi_1), (f_{n2}, \xi_2)\} \) is approximately normalized in \( \mathcal{H} \) in the sense that there exist positive constants \( c_1, c_2 \) independent of \( n \), such that

\[ c_1 \leq \|f_{ns}\|_{L^2(0,1)} = \sqrt{7} \|g_{ns}\|_{L^2(0,1)}, \quad |\xi_{ns}| \leq c_2, \quad n \in \mathbb{N}, \quad s = 1, 2, \]

**Proof.** We consider only the case \( |\tilde{k}_p| > 1 \), and the same arguments can be applied when \( |\tilde{k}_p| < 1 \). From (3.3), (3.5), (3.6), and linear algebra theory, the function \( g \), with respect to the eigenvalue \( \lambda = \frac{\alpha \rho^2}{\ell^2} - \frac{\beta^2}{4n^2} \), where \( \rho \in \mathbb{C} \) and \( \arg \rho \in [-\pi/2, \pi/2] \), is given by

\[ e^{-\frac{\ell}{2n}z} g(z, \rho) = e^{-\frac{\ell}{2n}z} \begin{vmatrix} e^{\tau_1 z} & e^{\tau_2 z} \\ 1 & \tau_2 e^{\tau_1 z} - \tau_1 e^{\frac{\ell}{2n}z} \end{vmatrix} = e^{\frac{\ell}{2n}z} e^{-\rho(1-z)} - \tau_1 e^{\frac{\ell}{2n}z} e^{\rho(1-z)}. \]

The first estimate of (3.24) is a consequence of (3.26) by setting

\[ g_{ns}(z) = -\rho_{ns}^{-1} e^{-\frac{\ell}{2n}z} g(z, \rho_{ns}), \quad s = 1, 2, \]

in (3.26). Regarding the second one, it can be proved as follows:

\[ \xi_{ns} = \frac{g_{ns}(0) - k_p g_{ns}(1)}{k_I} = \frac{2\tilde{k}_p - 2k_p e^{\frac{\ell}{2n}z}}{k_I} + O(n^{-1}) = O(n^{-1}), \]

where we have used the fact that

\[ e^{\rho_{ns}} + e^{-\rho_{ns}} - 2\tilde{k}_p = O(n^{-1}) \]

and the definition of \( \tilde{k}_p \) given in (3.8). Finally, in order to prove (3.25), we use (3.18) to obtain that \( \|e^{\frac{\ell}{2n}z} e^{\rho_{ns}(1-z)}\|^2_{L^2(0,1)} \) and \( \|e^{\frac{\ell}{2n}z} e^{-\rho_{ns}(1-z)}\|^2_{L^2(0,1)} \) are uniformly bounded in \( (0, 1) \). This, together with \( |\xi_{ns}|^2 = O(n^{-2}) \), gives

\[ \int_0^\ell |f_{ns}(x)|^2 dx = \ell \int_0^1 |g_{ns}(z)|^2 dz \]
and the desired inequalities (3.25) \( \square \)

Moreover, one can notice that the same process can also produce asymptotic expansions for the eigenpairs of \( A^*_PI \), which is the adjoint operator of \( A_PI \) and is given by

\[
(3.27) \quad A^*_PI \left( \begin{array}{c} h \\ \eta \end{array} \right) := \left( \begin{array}{c} \alpha h'' + \beta h' \\ \alpha k_I h'(0) \end{array} \right) \forall \left( \begin{array}{c} h \\ \eta \end{array} \right) \in \mathcal{D}(A^*_PI)
\]

and

\[
(3.28) \quad \mathcal{D}(A^*_PI) := \{(z, \eta) \in H^2(0, \ell) \times \mathbb{C} : h(0) = 0, \eta = \alpha h'(\ell) + \beta h(\ell) - \alpha k_P h'(0)\}.
\]

Since \( A_PI \) is a discrete operator, then so is \( A^*_PI \) (see [10, p. 2354]). Moreover, if \( \lambda \) is an eigenvalue of \( A_PI \), then \( \lambda \) is an eigenvalue of \( A^*_PI \) (see [20, p. 26]). Hence, we can get the eigenvalues of \( A^*_PI \) directly from (3.10) and (3.19) for \(|\bar{k}_P| > 1\) and \(|\bar{k}_P| < 1\), respectively, with the same algebraic multiplicity (see [10, p. 2354]). Also, the same arguments used in the proof in Theorem 3.4 will yield the counterpart of Theorem 3.4 for \( A^*_PI \), namely, the following.

**Theorem 3.5.** Let \( \sigma(A^*_PI) = \{\lambda_{n1} = \sqrt{\nu}, \lambda_{n2} = \sqrt{\nu}, n \in \mathbb{N}\} \) be the eigenvalues of \( A^*_PI \).

If \(|\bar{k}_P| > 1\) (respectively, \(|\bar{k}_P| < 1\)), \( \lambda_n = \frac{\alpha_2}{\alpha_3} - \frac{\beta_2}{\alpha_3} \), with \( \rho_n \in \mathbb{C} \) and \( \arg \rho_n \in [-\pi/2, \pi/2] \), are given by (3.10) and (3.18) (respectively, (3.19) and (3.23)). Furthermore, the corresponding eigenfunctions \( \{(h_{n1}, \eta_{n1}), (h_{n2}, \eta_{n2})\} \) have the following asymptotics: for \( s = 1, 2 \),

\[
(3.29) \quad \begin{cases} h_{ns}(z) = \phi_{ns}(z) = e^{-\frac{1}{2} \frac{\beta_2}{\alpha_3} z} e^{\rho_{ns} z} - e^{-\frac{1}{2} \frac{\beta_2}{\alpha_3} z} e^{-\rho_{ns} z} = e^{-\frac{1}{2} \frac{\beta_2}{\alpha_3} z} e^{\rho_{ns} z} - e^{-\frac{1}{2} \frac{\beta_2}{\alpha_3} z} e^{-\rho_{ns} z}, \\
\eta_{ns} = O(n^{-1})
\end{cases}
\]

for sufficiently large positive integer \( n \), where

\[
(3.30) \quad x = z \ell, \quad h(x) = \phi(z), \quad z \in [0, 1].
\]

Moreover, \( \{(h_{ns}, \eta_{ns}), s = 1, 2\} \) is approximately normalized in \( \mathcal{H} \).

**Proof.** From (3.27) and (3.28), the eigenvalue problem of \( A^*_PI \) is

\[
(3.31) \quad \begin{cases} h''(x) + \frac{\beta}{\alpha} h'(x) - \frac{1}{\alpha} \lambda h(x) = 0, & 0 < x < \ell, \\
h(0) = 0, \quad \alpha (\lambda k_P + k_I) h'(0) = \lambda \alpha h'(\ell) + \lambda \beta h(\ell).
\end{cases}
\]

By the transformation (3.30), which is similar to (3.2), the above equation changes into

\[
(3.32) \quad \begin{cases} \phi''(z) + \frac{\beta_2}{\alpha_3} \phi'(z) - \frac{\beta_2}{\alpha_3} \lambda \phi(z) = 0, & 0 < z < 1, \\
\phi(0) = 0, \quad \alpha (\lambda k_P + k_I) \phi'(0) = \lambda \alpha \phi'(1) + \lambda \beta \phi(1).
\end{cases}
\]

Then \( e^{-\tau_1 z} \) and \( e^{-\tau_2 z} \) are two independent solutions of \( \phi''(z) + \frac{\beta_2}{\alpha_3} \phi'(z) - \frac{\beta_2}{\alpha_3} \lambda \phi(z) = 0 \).

As in the proof of Theorem 3.4, the function \( \phi \) with respect to the eigenvalue \( \lambda \) of \( A^*_PI \), where \( \lambda = \frac{\alpha_2}{\alpha_3} - \frac{\beta_2}{\alpha_3} \), with \( \rho \in \mathbb{C} \) and \( \arg \rho \in [-\pi/2, \pi/2] \), is given by

\[
(3.33) \quad \phi(z, \rho) = \begin{vmatrix} 1 \\ e^{-\tau_1 z} \end{vmatrix} = e^{-\tau_2 z} - e^{-\tau_1 z} = e^{-\frac{1}{2} \frac{\beta_2}{\alpha_3} z} e^{\rho z} - e^{-\frac{1}{2} \frac{\beta_2}{\alpha_3} z} e^{-\rho z}.
\]
The first expression of (3.29) then follows from (3.33) by setting
\[ \phi_{ns}(z) = \phi(z, \overline{\nu_{ns}}), \quad s = 1, 2, \]
in (3.33), and \( \eta_{ns} \) can be obtained by
\[ \eta_{ns} = \alpha h_{ns}(\ell) + \beta h_{ns}(\ell) - \alpha k_p h_{ns}(0) = \frac{\alpha}{\ell} \phi_{ns}(1) + \beta \phi_{ns}(1) - \frac{\alpha}{\ell} k_p \phi_{ns}(0) = O(n^{-1}). \]
Here we have used the fact that
\[ e^{\overline{\nu_{ns}}} + e^{-\overline{\nu_{ns}}} - 2k_p = O(n^{-1}) \]
and the definition of \( \tilde{k}_p \) in (3.8). A direct computation can further show that \( \{(h_{ns}, \eta_{ns}), s = 1, 2\} \) is approximately normalized. \( \square \)

3.2. Completeness of the root subspace. First, we have the following.

**Theorem 3.6.** Let \( \sigma(A_{Pl}) = \{\lambda_{n1}, \lambda_{n2}, n \in \mathbb{N}\} \) be the eigenvalues of \( A_{Pl} \), and let \( \lambda_n = \frac{\alpha \rho^2}{\ell^2} - \frac{\beta^2}{4\alpha} \), with \( \rho_n \in \mathbb{C} \) and \( \arg \rho_n \in [-\pi/2, \pi/2) \). Then there exists a constant \( \tilde{M} > 0 \) independent of \( \lambda \) such that
\[ ||R(\lambda, A_{Pl})|| \leq \tilde{M} |\lambda|^{-1/2} \]
for all \( \lambda = \frac{\alpha \rho^2}{\ell^2} - \frac{\beta^2}{4\alpha}, \) with \( \rho \in \mathbb{C} \), and \( \arg \rho \in [-\pi/2, \pi/2) \) lies outside all circles of radius \( \varepsilon > 0 \) and the circles are centered at the zeros of \( \det(\Delta(\rho)) = 0 \) (see (3.9)).

**Proof.** Let \( \lambda = \alpha \rho^2/\ell^2 - \frac{1}{\alpha} \beta^2/\alpha \in \rho(A_{Pl}), \) with \( \rho \in \mathbb{C} \) and \( \arg \rho \in [-\pi/2, \pi/2), \) and let \( (\phi, c) \in H. \) Our aim is to solve the resolvent equation
\[ (\lambda I - A_{Pl}) \begin{pmatrix} f \\ \xi \end{pmatrix} = \begin{pmatrix} \phi \\ c \end{pmatrix}, \]
or, equivalently,
\[ \begin{cases} \lambda f(x) - \alpha f''(x) + \beta f'(x) = \phi(x), \\ \frac{\lambda f(0) - k_p f(\ell)}{k_1} - f(\ell) = c, \end{cases} \]
or
\[ \begin{cases} f''(x) - \frac{\beta}{\alpha} f'(x) - \frac{1}{\alpha} \lambda f(x) = -\frac{1}{\alpha} \phi(x), \quad 0 < x < \ell, \\ \lambda f(0) - \lambda k_p f(\ell) - k_1 f(\ell) = k_1 c, \quad f'(\ell) = 0. \end{cases} \]
By using the transformation (3.2), we obtain
\[ \begin{cases} g''(z) - \frac{\ell \beta}{\alpha} g'(z) - \frac{\ell}{\alpha} \lambda g(z) = -\frac{\ell}{\alpha} \phi(z \ell), \quad 0 < z < 1, \\ \lambda g(0) - (\lambda k_p + k_1) g(1) = k_1 c, \quad g'(1) = 0. \end{cases} \]
Set
\[ \Phi(z, \lambda) := g(z) - \frac{k_1 c}{\lambda(1 - k_p) - k_1}, \quad \lambda(1 - k_p) \neq k_1. \]
Thus we conclude from (3.9) that
\[
\frac{\partial^2}{\partial \rho^2} \Phi(z, \lambda) - \frac{\ell^2}{\alpha} \Phi'(z, \lambda) - \frac{\ell^2}{\alpha} \lambda \Phi(z, \lambda) = -\frac{\ell^2}{\alpha} \phi(z \ell) + \frac{\ell^2}{\alpha} \frac{\lambda k_I c}{\lambda(1-k_P) - k_I},
\]
\[
\lambda \Phi(0, \lambda) - (\lambda k_P + k_I) \Phi(1, \lambda) = 0, \quad \Phi'(1, \lambda) = 0.
\]
Therefore, every solution \( \Phi(z, \lambda) \) of (3.38) can be represented as (see, e.g., [32, p. 31, Theorem 2])
\[
\Phi(z, \lambda) = \int_0^1 G(z, \xi, \lambda) \left(-\frac{\ell^2}{\alpha} \phi(\xi \ell) + \frac{\ell^2}{\alpha} \frac{\lambda k_I c}{\lambda(1-k_P) - k_I}\right) d\xi,
\]
and hence the solution of (3.36) can be written as follows:
\[
g(z) = \int_0^1 G(z, \xi, \lambda) \left(-\frac{\ell^2}{\alpha} \phi(\xi \ell) + \frac{\ell^2}{\alpha} \frac{\lambda k_I c}{\lambda(1-k_P) - k_I}\right) d\xi + \frac{k_I c}{\lambda(1-k_P) - k_I}.
\]
Here \( G(z, \xi, \lambda) \) is the Green’s function given by
\[
G(z, \xi, \lambda) := \frac{1}{\det(\Delta(\lambda))} H(z, \xi, \lambda),
\]
with
\[
H(z, \xi, \lambda) := \begin{vmatrix} e^{\tau_1 z} & e^{\tau_2 z} & \eta(z, \xi, \lambda) \\ U_1(e^{\tau_1 z}) & U_1(e^{\tau_2 z}) & U_2(e^{\tau_1 z}) \\ U_2(e^{\tau_2 z}) & U_2(y_2) & U_2(\eta) \end{vmatrix},
\]
\[
\eta(z, \xi, \lambda) := \frac{1}{4\rho} \text{sign}(z - \xi) \left(e^{\tau_1(z - \xi)} - e^{\tau_2(z - \xi)}\right),
\]
where we have used the fact that \( \lambda = \frac{\alpha}{\ell^2} - \frac{\beta^2}{4\alpha}, \) \( U_1(g) := (\lambda k_P + k_I)g(1) - \lambda g(0), \) and \( U_2(g) := g'(1). \) Based on the above equations, we can claim that for \( \lambda \in \rho(\mathcal{A}_{PI}), \) with \( |\lambda| \) large enough, there exists a constant \( M \) independent of \( z, \xi \in [0, 1] \) so that
\[
|H(z, \xi, \lambda)| \leq M e^{\frac{\ell}{2\rho} |\rho|^2 |z|} \quad \rho \in \mathbb{C}, \quad \arg \rho \in [-\pi/2, \pi/2).
\]
Thus we conclude from (3.9) that
\[
|G(z, \xi, \lambda)| \leq M_1 |\rho|^{-1}
\]
holds for all \( \rho \in \mathbb{C}, \) with \( \arg \rho \in [-\pi/2, \pi/2) \) outside those circles of radius \( \varepsilon > 0 \) and centered at the zeros of \( \det(\Delta(\rho)) = 0, \) where \( M_1 \) is some constant independent of \( z, \xi \in [0, 1]. \) This will, in turn, yield estimates for \( g(z), f(x), \) and \( \|R(\lambda, \mathcal{A}_{PI})\|, \) respectively, as follows:
\[
|f(x)| = |g(z)| \leq M_2 |\rho|^{-1}
\]
and
\[
\|R(\lambda, \mathcal{A}_{PI})\| \leq M_3 |\rho|^{-1}
\]
for all \( \rho \in \mathbb{C} \), with \( \arg \rho \in [\pi/2, \pi) \) outside those circles of radius \( \varepsilon > 0 \) and centered at the zeros of \( \det(\Delta(\rho)) = 0 \), where \( M_2 \) and \( M_3 \) are some constants independent of \( z, \xi \in [0,1] \). This achieves the proof of the desired result by taking \( M = M_3 \).

Note that we can obtain a more precise estimate for \( \|R(\lambda, A_{\mathcal{P}I})\| \) as follows (for more details, the reader is referred to [14]).

**Theorem 3.7.** There exist positive constants \( r \) and \( M_r \), such that for an arbitrary \( \kappa \in (\pi/2, \pi) \), we have

\[
\Sigma_{\kappa,r} := \{ \lambda \in \mathbb{C} : |\arg(\lambda - r)| < \kappa, \lambda \neq 0 \} \subset \rho(\mathcal{A}_{\mathcal{P}I})
\]

and

\[
(3.47) \quad \|R(\lambda, \mathcal{A}_{\mathcal{P}I})\| \leq \frac{M_r}{|\lambda - r|} \quad \forall \lambda \in \Sigma_{\kappa,r}.
\]

We have the following lemma.

**Lemma 3.8.** Let \( \sigma(\mathcal{A}_{\mathcal{P}I}) = \{ \lambda_{n1}, \lambda_{n2}, n \in \mathbb{N} \} \) be the eigenvalues of \( \mathcal{A}_{\mathcal{P}I} \). If \( |\overline{k}_P| > 1 \) (respectively, \( 0 < |\overline{k}_P| < 1 \)), then \( \lambda_n = \frac{\alpha p^2}{\beta^2} - \frac{\beta^2}{4\alpha} \), with \( \rho_n \in \mathbb{C} \) and \( \arg \rho_n \in [-\pi/2, \pi/2) \), are given by (3.10) and (3.18) (respectively, (3.19) and (3.23)). In addition, for any sufficiently large \( n \), each eigenvalue \( \lambda_{ni}, i = 1, 2 \), of \( \mathcal{A}_{\mathcal{P}I} \) is algebraically simple.

**Proof.** From (3.40), the multiplicity of each \( \lambda \in \sigma(\mathcal{A}_{\mathcal{P}I}) \) with sufficiently large modulus, as a pole of \( R(\lambda, \mathcal{A}_{\mathcal{P}I}) \), is less than or equal to the multiplicity of \( \lambda \) as a zero of the entire function \( \det(\Delta(\rho)) \) with respect to \( \rho \). On the other hand, it is a routine exercise to verify that \( \lambda \) is geometrically simple. Since from (3.14), under the assumption \( |\overline{k}_P| \neq 1 \), all zeros of \( \det(\Delta(\rho)) = 0 \) with large moduli are simple, the result then follows from the general formula \( m_{\alpha} \leq p \cdot m_{g} \) (see, e.g., [30, p. 148]), where \( p \) denotes the order of the pole of the resolvent operator and \( m_{\alpha}, m_{g} \) denote, respectively, the algebraic and geometric multiplicity.

The next result follows.

**Proposition 3.9.** Both root subspaces of the operators \( \mathcal{A}_{\mathcal{P}I} \) and \( \mathcal{A}_{\mathcal{P}I}^* \) are complete in \( \mathcal{H} \), that is to say, \( \text{Sp}(\mathcal{A}_{\mathcal{P}I}) = \text{Sp}(\mathcal{A}_{\mathcal{P}I}^*) = \mathcal{H} \).

**Proof.** It follows from Lemma 5 on page 2355 of [10] that the following orthogonal decomposition holds:

\[
\mathcal{H} = \sigma_{\infty}(\mathcal{A}_{\mathcal{P}I}^*) \oplus \text{Sp}(\mathcal{A}_{\mathcal{P}I}),
\]

where \( \sigma_{\infty}(\mathcal{A}_{\mathcal{P}I}^*) \) consists of those \( Y \in \mathcal{H} \) so that \( R(\lambda, \mathcal{A}_{\mathcal{P}I}^*) Y \) is an analytic function of \( \lambda \) in the whole complex plane. Hence \( \text{Sp}(\mathcal{A}_{\mathcal{P}I}) = \mathcal{H} \) if and only if \( \sigma_{\infty}(\mathcal{A}_{\mathcal{P}I}^*) = \{ 0 \} \).

Now suppose that \( Y \in \sigma_{\infty}(\mathcal{A}_{\mathcal{P}I}^*) \). Since \( R(\lambda, \mathcal{A}_{\mathcal{P}I}^*) Y \) is an analytic function of \( \lambda \), it is also for \( \rho \), and hence by the maximum modulus principle (or the Phragmén-Lindelöf theorem) of analytic functions and the fact that \( \|R(\lambda, \mathcal{A}_{\mathcal{P}I}^*)\| = \|R(\overline{\lambda}, \mathcal{A}_{\mathcal{P}I})\| \), it follows from Theorem 3.6 that

\[
\|R(\lambda, \mathcal{A}_{\mathcal{P}I}^*) Y\| \leq M|\lambda|^{-1/2}\|Y\| \quad \forall \lambda \in \mathbb{C}
\]

for some constant \( M > 0 \). By Theorem 1 on page 3 of [17], we conclude that \( R(\lambda, \mathcal{A}_{\mathcal{P}I}^*) Y \) is a constant with respect to \( \lambda \), i.e.,

\[
R(\lambda, \mathcal{A}_{\mathcal{P}I}^*) Y = Y_0 \quad \text{for some } Y_0 \in \mathcal{H}.
\]

Thus

\[
Y = (\lambda - \mathcal{A}_{\mathcal{P}I}^*) Y_0 = -\mathcal{A}_{\mathcal{P}I} Y_0 + \lambda Y_0 \quad \forall \lambda \in \mathbb{C}.
\]
Finally, comparing the coefficients of $\lambda^j$, one can readily find that $Y_0 = 0$. This concludes the proof of the result. \hfill \square

### 3.3. Riesz basis generation

In order to establish the Riesz basis property of the root subspace of $A_{PI}$, we need the following lemma from [38].

**Lemma 3.10.** Suppose that a sequence $\{\nu_n\}$ has asymptotics

\begin{equation}
\nu_n = \alpha(n + i\beta \ln n) + O(1), \quad \alpha \neq 0, \quad n = 1, 2, 3, \ldots, \tag{3.48}
\end{equation}

where $\beta$ is a real number and $\sup_{n \geq 1} \text{Re} \nu_n < \infty$. Then the sequence $\{e^{\nu_n x}\}_{n=1}^{\infty}$ is a Bessel sequence in $L^2(0, 1)$.

Then, we have the following.

**Lemma 3.11.** Let $\rho_{ns}, s = 1, 2$, be given by (3.18) and (3.23) according to $|k| > 1$ and $|k| < 1$, respectively. Then $\{e^{\rho_{n1}}, e^{\rho_{n2}}\}_{n=1}^{\infty}$ and $\{e^{-\rho_{n1}}, e^{-\rho_{n2}}\}_{n=1}^{\infty}$ are two Bessel sequences in $L^2(0, 1)$. Hence, $\{e^{\rho_{n1}x}, e^{\rho_{n2}x}\}_{n=1}^{\infty}$ and $\{e^{-\rho_{n1}x}, e^{-\rho_{n2}x}\}_{n=1}^{\infty}$ are also two Bessel sequences in $L^2(0, \ell)$.

**Proof.** Let $\nu_{ns} := \rho_{ns}, s = 1, 2$. Then the result concerning the sequence $\{e^{\rho_{n1}}, e^{\rho_{n2}}\}_{n=1}^{\infty}$ can be directly obtained from Lemma 3.10 by setting $\beta = 0$ and $\alpha = 2\pi i$ in (3.48). Similarly, for the sequence $\{e^{-\rho_{n1}}, e^{-\rho_{n2}}\}_{n=1}^{\infty}$ one can take $\nu_n := -\rho_{ns}, s = 1, 2, \beta = 0$, and $\alpha = -2\pi i$ in (3.48). The last result is obvious since it can be directly verified from the definition of Bessel sequences (see [42, p. 122]). \hfill \square

Now, we are able to prove the Riesz basis property for the operator $A_{PI}$ stated in Theorem 2.4.

**Proof of Theorem 2.4.** Let $|k| \neq 1$ and let $\sigma(A_{PI}) = \{\lambda_{n1}, \lambda_{n2}, n \in \mathbb{N}\}$ be the eigenvalues of $A_{PI}$. From Lemma 3.8, we have that each eigenvalue of $A_{PI}$ with sufficient large modulus is simple, and hence there exists an integer $N \geq 0$ so that for all $n > N$, $\lambda_{ns}, s = 1, 2$, is simple. For $n \leq N$, if the algebraic multiplicity of each $\lambda_{ns}$ is $m_{ns}$, we can find the highest order generalized eigenfunction $\Phi_{n, s, 1}$ from

\begin{equation}
(A_{PI} - \lambda_{ns})^{m_{ns}} \Phi_{n, s, 1} = 0 \quad \text{but} \quad (A_{PI} - \lambda_{ns})^{m_{ns} - 1} \Phi_{n, s, 1} \neq 0, \quad s = 1, 2.
\end{equation}

The other lower order linearly independent generalized eigenfunctions associated with $\lambda_{ns}$ can be found through $\Phi_{n, s, j} = (A_{PI} - \lambda_{ns})^{j-1} \Phi_{n, s, 1}, j = 2, 3, \ldots, m_{ns}$. Assume $\Phi_{ns}$ is an eigenfunction of $A_{PI}$ corresponding to $\lambda_{ns}$ with $n > N$. Then $\{\{\Phi_{n, s, j}\}_{j=1}^{m_{ns}}\}_{n \leq N} \cup \{\Phi_{ns}\}_{n > N}$ are all linearly independent generalized eigenfunctions of $A_{PI}$. Let $\{\Psi_{n, s, j}\}_{j=1}^{m_{ns}}\}_{n \leq N} \cup \{\Psi_{ns}\}_{n > N}$ be the biorthogonal sequence of $\{\Phi_{n, s, j}\}_{j=1}^{m_{ns}}\}_{n \leq N} \cup \{\Phi_{ns}\}_{n > N}$. Then $\{\{\Psi_{n, s, j}\}_{j=1}^{m_{ns}}\}_{n \leq N} \cup \{\Psi_{ns}\}_{n > N}$ are all linearly independent generalized eigenfunctions of $A_{PI}$. It is well known that these two sequences are minimal in $H$, and from Proposition 3.9 they are also complete in $H$.

Hence, in order to prove the Riesz basis of the system, it suffices to show that both eigenfunctions $\{\Phi_{ns}\}_{n > N, s = 1, 2}$ and $\{\Psi_{ns}\}_{n > N, s = 1, 2}$ of the operators $A_{PI}$ and $A_{PI}^*$ are, respectively, Bessel sequences in $H$. Since $1 \leq \|\Phi_{ns}\| \|\Psi_{ns}\| \leq M$ for some constant $M$ independent of $n$ (see [42, p. 19]), we may assume without loss of generality that $\Phi_{ns} = (f_{ns}, \xi_{sn})$ given by (3.24) and $\Psi_{ns} = (h_{ns}, \eta_{ns})$ given by (3.29) for all $n > N$. Then it follows from Lemma 3.11 and the expansions (3.24) and (3.29) that both sequences $\{f_{n1}, f_{n2}\}_{n > N}$ and $\{h_{n1}, h_{n2}\}_{n > N}$ are Bessel sequences in $L^2(0, \ell)$ and both $\{\xi_{n1}, \xi_{n2}\}_{n > N}$ and $\{\eta_{n1}, \eta_{n2}\}_{n > N}$ are Bessel sequences in $C$. Therefore both of $\{\Phi_{ns}\}_{n > N, s = 1, 2}$ and $\{\Psi_{ns}\}_{n > N, s = 1, 2}$ are also Bessel sequences in $H$, and the result follows.
Now, let $|\bar{k}_P| = 1$. From (3.13), we already know that the eigenvalues $\lambda_{n1}$ and $\lambda_{n2}$ are not separable when their moduli are large enough, and hence the system may have the same eigenvalues depending on the feedback constants $k_P$ and $k_I$. For such a case, the generalized eigenfunctions of the operator $A_{PI}$ form, in $\mathcal{H}$, a Riesz basis with parentheses, and each eigenspace, corresponding to eigenvalues with enough large modulus, has dimension two [38]. We omit the details here and leave this as an exercise for the reader.

Finally, we reach the proof of the second main result of this work.

**Proof of Theorem 2.5.** We shall assume that $|\bar{k}_P| \neq 1$ (the case $|\bar{k}_P| = 1$ can be treated similarly). First, the existence of the $C_0$-semigroup $S_{PI}(t)$ follows from Lemma 3.8 and Theorem 2.4. Indeed, since $\{\{\Phi_{n,s,j}\}_{j=1}^{m_n}\}_{n,s=1,2} \cup \{\Phi_{ns}\}_{n>N,s=1,2}$ forms a Riesz basis for $\mathcal{H}$, then any $Y \in \mathcal{H}$ can be expanded as follows:

$$Y = \sum_{s=1}^{2} \sum_{n=1}^{N} \sum_{j=1}^{m_n} a_{nsj} \Phi_{n,s,j} + \sum_{s=1}^{2} \sum_{n=N+1}^{\infty} a_{ns} \Phi_{ns},$$

where $a_{nsj}$ and $a_{ns}$ are constants. Moreover, for such $Y$, the semigroup $S_{PI}(t)$ satisfies

$$S_{PI}(t)Y = \sum_{s=1}^{2} \sum_{n=1}^{N} \sum_{j=1}^{m_n} e^{\lambda_{ns} t} \sum_{i=0}^{m_n-j} \frac{t^i}{i!} a_{nsj} \Phi_{n,s,j} + \sum_{s=1}^{2} \sum_{n=N+1}^{\infty} e^{\lambda_{ns} t} a_{ns} \Phi_{ns}.$$  

(3.49)

Then, using the estimate (3.47) of $\|R(\lambda, A_{PI})\|$ on $\Sigma_{\kappa, r}$, one can deduce the analytic property of the semigroup (see [33]). Finally, the spectrum-determined growth condition is a direct consequence of the Riesz basis property (also from the analyticity of the semigroup).

**4. Proof of Theorem 2.6.** It is well known that although the spectrum of the uncontrolled operator $A_0$ lies in the open left half-plane (see Lemma 2.5), the operator $A_{PI}$ (see (2.10)–(2.11)) of the closed-loop system may have eigenvalues in the right half-complex-plane, and hence the semigroup $S_{PI}(t)$ will be unstable if the proportional and/or integral gains $k_P$ and/or $k_I$ are not properly chosen. We thus need to propose a design method for the proportional and integral gains $k_P$ and $k_I$ so that the closed-loop system (2.9) will be exponentially stable.

Using Theorem 4.3 of [33], one can claim that in order to get the exponential stability of the analytic semigroup $S_{PI}(t)$, it suffices to show that all the eigenvalues of the operator $A_{PI}$ defined by (2.10)–(2.11) have negative real part. Unfortunately, this property turns out to be very difficult to establish in our case since (i) it is not obvious to prove the dissipativity of the operator $A_{PI}$ even if under conditions on the proportional and integral gains $k_P$ and $k_I$, and (ii) the problem of obtaining an explicit expression of the eigenvalues of $A_{PI}$ is equivalent to solving an unusual transcendental equation. Therefore, it is not easy to get an explicit condition on the proportional and integral gains $k_P$ and $k_I$, which involves the system parameters $\alpha$, $\beta$, and $\ell$ so that the eigenvalues lie in the open left half-plane. In return, we are able to give some implicit conditions on $k_P$ and $k_I$ to resolve this hard problem.

First, we have the following result.

**Proposition 4.1.** If the proportional and integral gains $k_P$ and $k_I$ are negative, then any real eigenvalue of the operator $A_{PI}$, defined by (2.10)–(2.11), must be necessarily negative.
Proof. Let \( \lambda \) be an eigenvalue of \( A_{PI} \) and \( \phi = (f, \xi) \) an associated eigenfunction. Then its eigenvalue problem (3.1) has two characteristic roots given by

\[
\eta_{1,2} = \frac{\beta \pm \sqrt{\Delta_\lambda}}{2\alpha}, \quad \text{where} \quad \Delta_\lambda = \beta^2 + 4\alpha \lambda.
\]

Consider the following two cases.

Case (i). \( \eta_1 = \eta_2 \). This implies that \( \Delta_\lambda = 0 \), i.e., \( \lambda = -\frac{\beta^2}{4\alpha} \). Next, one can readily prove that \( \lambda = -\frac{\beta^2}{4\alpha} \) is an eigenvalue if and only if

\[
\frac{\beta^2}{4\alpha} k_P - k_I = \frac{\beta^2}{4\alpha} \left( 1 + \frac{\beta \ell}{2\alpha} \right) e^{-\frac{\beta \ell}{2\alpha}}.
\]

Hence for appropriate negative proportional and integral gains, \( \lambda = -\frac{\beta^2}{4\alpha} < 0 \) is an eigenvalue.

Case (ii). \( \eta_1 \neq \eta_2 \). In this case, let us consider the normalized solution, of (3.1), at \( x = \ell \), by \( f(\ell) = 1 \). This, together with the boundary condition \( f'(\ell) = 0 \), implies that the solution of (3.1) is

\[
(4.1) \quad f(x) = e^{-\frac{\beta x}{2\alpha \sqrt{\Delta_\lambda}}} \left[ \left( \beta + \sqrt{\Delta_\lambda} \right) e^{\eta_1 L + \eta_2 x} - \left( \beta - \sqrt{\Delta_\lambda} \right) e^{\eta_2 L + \eta_1 x} \right].
\]

Now let \( k_P, k_I < 0 \) and assume that \( \lambda = \gamma^2 \), where \( \gamma \in \mathbb{R} \setminus \{0\} \), is an eigenvalue of \( A_{PI} \). This is equivalent to claiming that the other boundary condition \( (\lambda k_P + k_I) = \lambda f(0) \) holds. In return, using (4.1) and the fact that \( f(\ell) = 1 \), it follows after a simple calculation that the following equation holds:

\[
\left( \beta + \sqrt{\Delta(\gamma)} \right) e^{\frac{\sqrt{\Delta(\gamma)} \ell}{2\alpha}} - \left( \beta - \sqrt{\Delta(\gamma)} \right) e^{-\frac{\sqrt{\Delta(\gamma)} \ell}{2\alpha}} = 2 e^{\frac{\beta \ell}{2\alpha}} \sqrt{\Delta(\gamma)} \left( k_P \gamma^2 + k_I \right),
\]

where \( \Delta(\gamma) = \beta^2 + 4\alpha \gamma^2 \). Finally, since \( \sqrt{\Delta(\gamma)} > \beta \) and \( k_P, k_I < 0 \), the above equation leads to a contradiction.

Let us rewrite the spectral system (3.1) in the following form:

\[
(4.2) \quad \begin{cases} 
\alpha f''(x, \lambda) - \beta f'(x, \lambda) = \lambda f(x, \lambda), & 0 \leq x \leq \ell, \\
\lambda f(0, \lambda) = (\lambda k_P + k_I) f(\ell, \lambda), & f'(\ell, \lambda) = 0.
\end{cases}
\]

Then, by considering the normalized solution at \( x = \ell \) via \( f(\ell, \lambda) = 1 \), it follows that all eigenvalues of (4.2) are the roots of the left-end boundary condition \( \lambda f(0, \lambda) = \lambda k_P + k_I \). Since \( f(\cdot, \lambda) \) satisfies the same equation, as in (2.3) and (2.6), it necessarily has the same representation given by (2.7), and hence the condition \( \lambda f(0, \lambda) = \lambda k_P + k_I \) yields

\[
(4.3) \quad B(\lambda) := \lambda \prod_{i=1}^{\infty} \left( 1 - \frac{\lambda}{\lambda_i^2} \right) = \lambda k_P + k_I.
\]

Recall that we have to exclude the zero solution \( \lambda = 0 \) of the above equation as \( 0 \in \rho(A_{PI}) \), the resolvent set of \( A_{PI} \). We are now able to prove the third main result stated as Theorem 2.6.

Proof of Theorem 2.6. (i) First, one can claim from the definition of \( B(\lambda) \) that \( B(\lambda) > 0 \) whenever \( \lambda \) is a positive real number, and hence any real root of \( B(\lambda) \) should
be negative. Second, since both $\lambda_i^0$ and $k_I$ are negative and $k_P$ is positive, it follows that $B(\lambda_i^0) \neq k_P\lambda_i^0 + k_I$, and thus for $n$ large enough, the zeros of $B(\cdot) - ((\cdot) k_P + k_I)$ and $B_n(\cdot) - ((\cdot) k_P + k_I)$ coincide, in the finite strip $\lambda_i^0 \leq \Re(z) \leq 0$ and $|\Im(z)| \leq \varepsilon$ which contains a piece of the negative real axis.

Now if $m_1 < k_I < 0$ and $k_P$ is a positive number sufficiently small (see Figure 2), then the $n + 1$ roots of $B_n(\lambda) = \lambda k_P + k_I$ are all real and negative. Finally, note that in this case, we may have $n - 1$ negative distinct real roots and one double negative real root (see Figure 3). Since these properties are valid for all $B_n$, they also hold true for $B$ by Lemma 2.3.

(ii) Consider now a large enough strip that would include the points where $B_n$ attains the minima $m_1$ and $m_2$ (see Figure 4). Then using the graph of $B_n$, one can see that if $m_2 < k_I < m_1$ and $k_P$ is a positive number sufficiently small, the equation $B_n(\lambda) = \lambda k_P + k_I$ has only $n - 1$ real roots instead of $n + 1$. Hence two roots are missing. Since $B_n(\lambda) = \lambda k_P + k_I$ must have $n + 1$ roots, the two missing roots should
go conjugate complex (which is a bifurcation) with positive or negative real part. These properties still remain true for $B$, by Lemma 2.3.

Finally, we are going to show the output regulation and the exponential stability of the closed-loop system (1.1)–(1.2). Although the proof is similar to that of Theorem 4.1 in [2], we would rather give some details for sake of completeness.

Due to Theorem 2.5 and the exponential stability of the semigroup $S_{PI}(t)$, the solution $\phi(t) = (Q(\cdot, t), \xi(t))$ of the closed-loop system (2.9), or alternatively (1.1)–(1.2), stemming from the initial data $\phi_0 = (Q_0, \xi_0) \in D(A_{PI})$, can be written as

$$
\phi(t) = S_{PI}(t)\phi_0 + \int_0^t S_{PI}(t-s)(w, -y_r)ds = S_{PI}(t)\phi_0 + A_{PI}^{-1}(S_{PI}(t) - I)(w, -y_r)
$$

and satisfies

$$(4.4) \quad \lim_{t \to \infty} \phi(t) = \lim_{t \to \infty} (Q(\cdot, t), \xi(t)) = -A_{PI}^{-1}(w, -y_r).$$

Let $\phi^* := (Q^*, y_r^*) = -A_{PI}^{-1}(w, -y_r)$ and hence $A_{PI}(Q^*, y_r^*) = (-w, y_r)$. Using (2.10)–(2.11), it follows that $Q^*(\ell) = y_r$. This, together with (4.4), implies that $\lim_{t \to -\infty} y(t) = \lim_{t \to -\infty} Q(t, t) = y_r$. Concerning the stability of the closed-loop system (2.9), we first note that the steady state is $\phi^* = A_{PI}^{-1}(-w, y_r)$. Then, setting $\tilde{\phi} := \phi - \phi^* = \phi - A_{PI}^{-1}(-w, y_r)$, the closed-loop system (2.9) can be written as

$$(4.5) \quad \dot{\tilde{\phi}}(t) = A_{PI}(\tilde{\phi}(t) + \phi^*) + (w, -y_r) = A_{PI}\tilde{\phi}(t) + A_{PI}\phi^* + (w, -y_r) = A_{PI}\tilde{\phi}(t).$$

Consequently, the exponential stability of the closed-loop system (2.9) is equivalent to that of the semigroup $S_{PI}(t)$, independently of the perturbation $w$. The proof is complete. \[ \square \]

**Remark 4.1.**

1. As mentioned in the introduction, one of the advantages of introducing the proportional gain $k_P$ in our control feedback law (contrary to the work [2] where $k_P = 0$) is that this gain improves the stability and the regulation of the closed-loop system (1.1)–(1.2), where $k_P \neq 0$, in comparison with the
closed-loop system (1.1)–(1.2) with \( k_P = 0 \). To see how this goes, let us sketch the argument from the proof of Theorem 2.6. Suppose that \( k_P = 0 \). Then if \( m_1 < k_I < 0 \) (the other cases can be treated similarly), the equation \( B_n(\lambda) = k_I \) has \( n + 1 \) distinct negative real roots (see Figure 5), namely, \( \mu_i, i = 1, \ldots, n \). Now, one can always choose \( k_P \) sufficiently small such that the new equation \( B_n(\lambda) = \lambda k_P + k_I \) has not only \( n + 1 \) distinct negative real roots \( \eta_i, i = 1, \ldots, n \), but also \( \max_i{\eta_i} < \max_i{\mu_i} \), and hence the spectrum is moved to the left as shown in Figure 5.

2. According to Proposition 4.1, one can claim that in order to recover all negative real eigenvalues of the operator \( A_{PI} \), it suffices to take both \( k_P \) and \( k_I \) negative and then to try adding further conditions on \( k_P \) and \( k_I \) to conclude the stability. Unfortunately, we have tried in this direction but without much success. This is due to the fact that when \( k_P \) and \( k_I \) are negative, the arguments of the proof of Theorem 2.6 fail. Indeed, one can easily check that for \( k_P, k_I < 0 \) and for given \( \alpha, \beta, \) and \( \ell \), the spectrum \( \sigma(A_{PI}) \) may contain many complex eigenvalues, and we cannot control their real parts by means of our approach. For instance, consider a river whose characteristics are \( \ell = 2700, \alpha = 2000, \) and \( \beta = 0.9 \) (see [31] for more details about this model). Then, let \( k_I = -0.01 \) and \( k_P = -0.05 \). Using MAPLE, one can verify that \( \lambda = 0.001150641022 + i0.003238193679 \) is an eigenvalue among others. This physical example shows that for \( k_P < 0 \), we may have complex eigenvalues with positive real part, and hence the closed-loop system is unstable. However, it is clear that this does not mean that the system is unstable whenever \( k_P < 0 \), but as mentioned above, this is a drawback of our approach. This is why we had to choose \( k_I < 0 \) and \( k_P > 0 \) in Theorem 2.6 to overcome this difficulty and conclude some results on the spectrum \( \sigma(A_{PI}) \).

5. Numerical applications. Consider a river reach (1.1)–(1.2) with length \( \ell = 2000m \) and a reference discharge \( Q_0 = 2m^3/s \). The coefficients are hence \( \alpha = 664 m^2/s \) and \( \beta = 7.7854 m/s \) (for more details about the model, the reader is referred to [18]). Now, assume the constant perturbation \( w = 1/2 \) and the reference \( y_r = 1 \). Next, we apply the finite difference method for the space variable \( x \)
to transform the distributed parameter system (1.1)–(1.2) to a first order system of differential equations and then use MATLAB. Taking the proportional gain $k_P = 0$ and the integral gain $k_I = -0.01$, we observe that the system spectrum consists of negative real numbers, and the output $y(t) = Q(L, t)$ is regulated to $y_r = 1$ (see Figure 6). Furthermore, we notice that for sufficiently small values of $k_P$, the spectrum moves to the left in the sense that the first eigenvalue of the system is shifted to the left (see the third parts of Figures 6 and 7) and thus the regulation is guaranteed with smaller values of time $t$ (see Figures 6 and 7). In other words, less time is needed to regulate the system, and hence both exponential stability and regulation are sped up. For $k_P = 0.259$, one complex conjugate pair, with negative real part, appears but we still have the stability as well as the regulation of the system (see Figure 8). However, although $k_I$ is unchanged, i.e., $k_I = -0.01$, if the proportional gain $k_P$ is not “sufficiently small,” for instance, $k_P = 0.9$, there are two complex eigenvalues with positive real part, and hence neither the stability nor the output regulation is guaranteed (see Figure 9).
Fig. 7. Negative real spectrum: Stability and output regulation for \( k_I = -0.01, k_P = 0.25825 \).
Fig. 8. One complex conjugate pair of eigenvalues with negative real part: Stability and output regulation for $k_I = -0.01$, $k_P = 0.259$.

Fig. 9. One complex conjugate pair of eigenvalues with positive real part: Nonregulation for $k_I = -0.01$, $k_P = 0.9$. 
Acknowledgments. The authors are grateful to the associate editor and the referees for their constructive criticism and valuable suggestions for improving the paper.

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