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Dynamic stabilization of an Euler–Bernoulli beam under boundary control and non-collocated observation[☆]

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Abstract

We study the dynamic stabilization of an Euler–Bernoulli beam system using boundary force control at the free end and bending strain observation at the clamped end. We construct an infinite-dimensional observer to track the state exponentially. A proportional output feedback control based on the estimated state is designed. The closed-loop system is shown to be non-dissipative but admits a set of generalized eigenfunctions, which forms a Riesz basis for the state space. As consequences, both the spectrum-determined growth condition and exponential stability are concluded.

Keywords: Euler–Bernoulli beam; Observer; Riesz basis; Controllability and observability; Stability

1. Introduction

Most approaches to controller design for systems described by partial differential equations (PDEs) use static collocated feedback. This means that the actuators and sensors are located in the same areas and are such that the control operator is the adjoint of the observation operator. Lyapunov or energy multiplier methods are often used for the stability analysis of the resulting closed-loop systems. It has been known for a long time that the performance of these closed-loop systems may not be good, see [4]. Several articles considered non-collocated control for specific systems described by PDEs using simulation and experiments [3,15,18,21] but mathematical investigations of such controllers are quite few. The first difficulty behind is that the open-loop form of a non-collocated systems is usually not minimum-phase. This results in the closed-loop system unstable under a small increment of large feedback controller gains. Secondly, the closed loop of a non-collocate system is usually non-dissipative. The well-posedness of non-dissipative systems is a big challenge and furthermore, the traditional Lyapunov function or the energy multiplier methods are not easy to be used for the analysis of the stability of the non-dissipative systems.

The first effort on stabilizing compensators for infinite-dimensional systems was made in [8] where a generalization of the Luenberger compensator was given for the system in which both input and output operators are bounded. The earlier results about compensators for the distributed parameter systems can be found in [1]. For finite-dimensional compensators with unbounded input and bounded/unbounded output operators systems, we refer to [6,14]. The abstract formulation for an observer-based stabilizing controller of regular well-posed linear infinite-dimensional systems was established in [23]. In past several years, some efforts are particularly made for flexible arms. The passivity property of a non-collocated single-link flexible manipulator with a parameterized output was studied in [15]. It was showed that for a non-collocated truncated passive transfer function, a PD controller is sufficient to stabilize the overall system [13] defined two transmission zero condition numbers that quantify zero sensitivity precisely

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and lead to a clear geometrical interpretation of the circumstances under which a non-collocated structure will posses ill-conditioned zeros. Recently, the estimated state feedbacks are designed through backstepping observers in [20] to stabilize a class of one-dimensional parabolic PDEs. The abstract observers design for a class of well-posed regular infinite-dimensional systems can be found in [7] but the stabilization was not addressed.

The objective of this paper is to generalize our recent work on the stabilization of wave equation under boundary control with non-collocated observation to the Euler–Bernoulli beam equation [9]. But there are some differences for these two kinds of systems. First, recent study shows that the stabilization of Euler–Bernoulli beam equations by backstepping methods is much hard than that for wave equations. It could be done only for some special boundary conditions through the control of linearized Schrödinger equations. Secondly, in some cases, Lyapunov methods are not effective in proving the stability for non-collocated beam equation but are effective for most of wave equations.

The problem we are concerned with is the following Euler–Bernoulli beam equation with boundary control and non-collocated observation:

$$\begin{cases} w_{tt}(x,t) + w_{xxxx}(x,t) = 0, \ 0 < x < 1, t > 0, \\ w(0,t) = w_x(0,t) = w_{xx}(1,t) = 0, \ t \geq 0, \\ w_{xxx}(1,t) = u(t), \ y(t) = w_{xx}(0,t), \ t \geq 0, \end{cases} \tag{1.1}$$

where u is the boundary shear control (or input) at $x = 1$ and the observation (or output) y is the bending strain at $x = 0$. This problem comes from a dynamic model of vibrating beam for which one end is clamped to a control motor shaft and rotated by the motor at an angular velocity $\dot{\theta}(t)$ in the horizontal plane ([16], pp. 176):

$$\begin{cases} z_{tt}(x,t) + z_{xxxx}(x,t) = -x\ddot{\theta}(t), \ 0 < x < 1, t > 0, \\ z(0,t) = z_x(0,t) = z_{xx}(1,t) = -z_{xxx}(1,t) = 0, \ t \geq 0, \\ y_z(t) = z_{tt}(1,t), \ t \geq 0, \end{cases} \tag{1.2}$$

where $y_z(t)$ is the acceleration point output of the system. It is motivated by the acceleration point-sensing problem for weakly coupled wave equation considered in [2]. Let $w(x,t) = z_{xx}(1-x,t), \ddot{\theta}(t) = u(t), y(t) = -u(t) - y_z(t)$; we get (1.1) from (1.2).

We consider the system (1.1) in the energy state space $\mathbb{H} = H_E^2(0,1) \times L^2(0,1), H_E^2(0,1) = \{f \mid f \in H^2(0,1), f(0) = f'(0) = 0\}$. $\mathbb{H}$ is equipped with the obvious inner product induced norm $\|(f,g)\|_{\mathbb{H}}^2 = \int_0^1 [|f''(x)|^2 + |g(x)|^2]\,\mathrm{d}x$ for any $(f,g) \in \mathbb{H}$. And the input (output) space is $U = \mathbb{C}^1$. Define a linear operator $\mathbb{A} : D(\mathbb{A}) (\subset \mathbb{H}) \to \mathbb{H}$ as following:

$$\mathbb{A}(f,g) = (g, -f^{(4)}), \quad D(\mathbb{A}) = \{(f,g) \in \mathbb{H} | \mathbb{A}(f,g) \in \mathbb{H}, \ f''(1) = f'''(1) = 0\}. \tag{1.3}$$

Then system (1.1) can be written as

$$\sum(\mathbb{A},\mathbb{B},\mathbb{C}) : \begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} w \\ w_t \end{pmatrix} = \mathbb{A}\begin{pmatrix} w \\ w_t \end{pmatrix} + \mathbb{B}u(t), \\ \mathbb{B} = \begin{pmatrix} 0 \\ -\delta(x-1) \end{pmatrix}, \\ y(t) = \mathbb{C}\begin{pmatrix} w \\ w_t \end{pmatrix} = w_{xx}(0,t), \\ \mathbb{C} = \left(\langle \delta''(x), \cdot \rangle, \ 0\right), \end{cases} \tag{1.4}$$

where $\delta(\cdot)$ denotes the Dirac delta function. Obviously, both $\mathbb{B}$ and $\mathbb{C}$ are unbounded operators.

Theorem 1.1. *For each $u \in L^2_{\mathrm{loc}}(0,\infty)$ and initial datum $(w(\cdot,0), w_t(\cdot,0)) \in \mathbb{H}$, there exists a unique solution $(w, w_t) \in C(0,\infty;\mathbb{H})$ to Eq. (1.1), and for each $T > 0$, there exists a $C_T > 0$ independent of u and $(w(\cdot,0), w_t(\cdot,0))$ such that*

$$\|(w(\cdot,T), w_t(\cdot,T))\|_{\mathbb{H}}^2 + \int_0^T |y(\tau)|^2 \mathrm{d}\tau \leq C_T \left[\|(w(\cdot,0), w_t(\cdot,0))\|_{\mathbb{H}}^2 + \int_0^T |u(\tau)|^2 \mathrm{d}\tau\right].$$

Proof. By the well-posed linear infinite-dimensional system theory [5,12], it is equivalent to showing that $\mathbb{C}$ is admissible for $\mathrm{e}^{\mathbb{A}t}$, $\mathbb{B}^*$ is admissible for $\mathrm{e}^{\mathbb{A}^*t}$ and the input–output map is bounded (see Definition 2.1, 2.5 of [12]). The former two facts are almost trivial, we need only to consider the boundedness of the input–output map under the zero initial condition: $w(x,0) = w_t(x,0) \equiv 0$.

Let $F(t) = \frac{1}{2}\int_0^1 [w_t^2(x,t) + w_{xx}^2(x,t)]\,\mathrm{d}x$. Then for any $T > 0$, we have $\dot{F}(t) = -w_t(1,t)u(t)$ and hence

$$F(t) \leq \frac{\delta}{2}\int_0^T w_t^2(1,t)\,\mathrm{d}t + \frac{1}{2\delta}\int_0^T u^2(t)\,\mathrm{d}t, \quad \forall\, t \in [0,T] \text{ and } \delta > 0. \tag{1.5}$$

Next, let $\rho_1(t) = \int_0^1 x w_x(x,t) w_t(x,t)\,\mathrm{d}x$. Then $|\rho_1(t)| \leq F(t)$ and

$$\dot{\rho}_1(t) = \frac{1}{2}w_t^2(1,t) - w_x(1,t)u(t) - \frac{1}{2}\int_0^1 [w_t^2(x,t) + 3w_{xx}^2(x,t)]\,\mathrm{d}x.$$

Integrate over $[0,T]$ with respect to t and make use of (1.5) to give

$$\begin{aligned} \frac{1}{2}\int_0^T w_t^2(1,t)\,\mathrm{d}t &\leq F(T) + \frac{1}{2}\int_0^T u^2(t)\,\mathrm{d}t + 4\int_0^T F(t)\,\mathrm{d}t \\ &\leq \left(\frac{\delta}{2} + 2\delta T\right)\int_0^T w_t^2(1,t)\,\mathrm{d}t \\ &\quad + \left(\frac{1}{2\delta} + \frac{1}{2} + \frac{2T}{\delta}\right)\int_0^T u^2(t)\,\mathrm{d}t \end{aligned} \tag{1.6}$$

and hence

$$\int_0^T w_t^2(1,t)\,dt \le \frac{2}{1-(1+4T)\delta}\left(\frac{1}{2\delta}+\frac{1}{2}+\frac{2T}{\delta}\right) \times \int_0^T u^2(t)\,dt \tag{1.7}$$

as $0<\delta<\frac{1}{1+4T}$. This together with (1.5) gives

$$F(t) \le C_{\delta,T}\int_0^T u^2(t)\,dt, \quad \forall\, t\in[0,T] \text{ and } 0<\delta<\frac{1}{1+4T}, \tag{1.8}$$

where $C_{\delta,T}$ is a constant depending on δ, T only.

Now, let $\rho_2(t) = \int_0^1 (x-1)w_x(x,t)w_t(x,t)\,dx$. Then $|\rho_2(t)| \le F(t)$ and

$$\dot\rho_2(t) = \frac{1}{2}w_{xx}^2(0,t) - \frac{1}{2}\int_0^1 [w_t^2(x,t)+3w_{xx}^2(x,t)]\,dx.$$

Integrate over $[0,T]$ with respect to t and make use of (1.8) to produce

$$\frac{1}{2}\int_0^T y^2(t)\,dt \le F(t) + 3\int_0^T F(t)\,dt \le (C_{\delta,T}+3TC_{\delta,T}) \times \int_0^T u^2(t)\,dt, \tag{1.9}$$

where $T>0$ and $0<\delta<\frac{1}{1+4T}$. The proof is complete. □

The significance of Theorem 1.1 is that it not only gives the well-posedness of the open-loop system (1.1) but also shows that for any L^2 control, the output y makes sense and is also in L^2. This fact is a key point to the design of the observer because for the observer, y becomes input and the L^2 property of y plays an important role in solvability of the observer. The remaining part of this paper is organized as follows. In Section 2, we construct an observer for the system (1.1) and show that this observer is exponentially convergent. Section 3 is devoted to the output feedback design via the estimated state. In Section 4, we analyze the asymptotic behavior of the eigenvalues. The Riesz basis and exponential stability are presented in Section 5. Some concluding remarks are given in last Section 6.

2. Observer design

We design the observer for the system (1.1) as:

$$\begin{cases} \widehat{w}_{tt}(x,t)+\widehat{w}_{xxxx}(x,t)=0, & 0<x<1, t>0,\\ \widehat{w}(0,t)=\widehat{w}_{xx}(1,t)=0, & \widehat{w}_{xxx}(1,t)=u(t), \ t\ge 0,\\ \widehat{w}_{xx}(0,t)=\alpha\widehat{w}_{xt}(0,t)+\beta\widehat{w}_x(0,t)+y(t), & t\ge 0, \end{cases} \tag{2.1}$$

where $\alpha,\beta>0$ are constants. It is seen that for observer (2.1), its inputs are the input and output of the system (1.1) that belong to L^2 by Theorem 1.1. First, we have to study the solvability of system (2.1). The system (2.1) is considered in the space $\mathbf{H} = H_L^2(0,1)\times L^2(0,1)$, $H_L^2(0,1) = \{f \mid f\in H^2(0,1), f(0)=0\}$ which is larger than $\mathbb{H}$. The inner product induced norm of $\mathbf{H}$ is given by

$$\|(p,q)\|^2 = \int_0^1 [|p''(x)|^2+|q(x)|^2]\,dx + \beta|p'(0)|^2, \ \forall\,(p,q)\in\mathbf{H}.$$

Define the operator $\mathbf{A}: D(\mathbf{A})(\subset\mathbf{H})\to\mathbf{H}$:

$$\begin{cases} \mathbf{A}(f,g) = (g,-f^{(4)}), \ \forall\,(f,g)\in D(\mathbf{A}),\\ D(\mathbf{A}) = \{(f,g)\in\mathbf{H}|\mathbf{A}(f,g)\in\mathbf{H},\\ \quad f''(0)=\alpha g'(0)+\beta f'(0),\ f''(1)=f'''(1)=0\}. \end{cases} \tag{2.2}$$

It is readily found that

$$\begin{cases} \mathbf{A}^*(\phi,\psi) = (-\psi,\phi^{(4)}), \ \forall\,(\phi,\psi)\in D(\mathbf{A}^*),\\ D(\mathbf{A}^*) = \{(\phi,\psi)\in\mathbf{H}|\ \mathbf{A}^*(\phi,\psi)\in\mathbf{H},\\ \quad \phi''(0) = -\alpha\psi'(0)+\beta\phi'(0),\\ \quad \phi''(1)=\phi'''(1)=0\}. \end{cases} \tag{2.3}$$

Take the inner product of $(\phi,\psi)\in D(\mathbf{A}^*)$ with Eq. (2.1) to obtain

$$\frac{d}{dt}\left\langle \begin{pmatrix}\widehat{w}\\ \widehat{w}_t\end{pmatrix}, \begin{pmatrix}\phi\\ \psi\end{pmatrix}\right\rangle = \left\langle \begin{pmatrix}\widehat{w}\\ \widehat{w}_t\end{pmatrix}, \mathbf{A}^*\begin{pmatrix}\phi\\ \psi\end{pmatrix}\right\rangle + \left\langle \begin{pmatrix}0\\ -\delta(x-1)\end{pmatrix}u(t), \begin{pmatrix}\phi\\ \psi\end{pmatrix}\right\rangle + \left\langle \begin{pmatrix}0\\ -\delta'(x)\end{pmatrix}w_{xx}(0,t), \begin{pmatrix}\phi\\ \psi\end{pmatrix}\right\rangle.$$

Hence, the system (2.1) can be written as

$$\frac{d}{dt}\begin{pmatrix}\widehat{w}\\ \widehat{w}_t\end{pmatrix} = \mathbf{A}\begin{pmatrix}\widehat{w}\\ \widehat{w}_t\end{pmatrix} + \mathbf{B}\begin{pmatrix}u\\ y(t)\end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix}0 & 0\\ -\delta(x-1) & -\delta'(x)\end{pmatrix}. \tag{2.4}$$

The following Theorem 2.1 assures the solvability of the observer.

Theorem 2.1. *For any $u(\cdot), y(\cdot)\in L^2_{loc}(0,\infty)$ and initial datum $(\widehat{w}(\cdot,0),\widehat{w}_t(\cdot,0))\in\mathbf{H}$, there exists a unique solution $(\widehat{w},\widehat{w}_t)\in C(0,\infty;\mathbf{H})$ to Eq. (2.1), and for all $T>0$, there exists a $D_T>0$ depending on T only such that*

$$\|(\widehat{w}(\cdot,T),\widehat{w}_t(\cdot,T))\|_{\mathbf{H}}^2 \le D_T\left\{\|(\widehat{w}(\cdot,0),\widehat{w}_t(\cdot,0))\|_{\mathbf{H}}^2 + \int_0^T \left[|u(\tau)|^2+|y(\tau)|^2\right]d\tau\right\}. \tag{2.5}$$

Proof. Again, it suffices to show that $\mathbf{B}^*$ is admissible for $e^{\mathbf{A}^*t}$ (see Definition 2.1 of [12] and Theorem 6.9 of [22]). A simple computation shows that this is equivalent to saying that $\mathbf{B}^*\mathbf{A}^{*-1}$ is bounded and for any $T>0$, and there exists an $M_T>0$ depending on T only such that the system of the following

$$\begin{cases} \widehat{w}_{tt}(x,t)+\widehat{w}_{xxxx}(x,t)=0, & 0<x<1, t>0,\\ \widehat{w}_{xx}(0,t)=\alpha\widehat{w}_{xt}(0,t)+\beta\widehat{w}_x(0,t), & t\ge 0,\\ \widehat{w}(0,t)=\widehat{w}_{xx}(1,t)=\widehat{w}_{xxx}(1,t)=0, & t\ge 0,\\ y_w(t) = (\widehat{w}_t(1,t),\ \widehat{w}_{xt}(0,t)), & t\ge 0 \end{cases} \tag{2.6}$$

satisfies (see also (2.3) of [19])

$$\int_0^T [|\widehat{w}_t(1,t)|^2 + |\widehat{w}_{xt}(0,t)|^2]dt \le M_T E_m(0),$$

where $E_m(t) = \frac{1}{2}\int_0^1 [|\widehat{w}_{xx}(x,t)|^2 + |\widehat{w}_t(x,t)|^2]\,dx + \frac{\beta}{2}|\widehat{w}_x(0,t)|^2$. First, a simple computation shows that

$$\mathbf{A}^{*-1}(\phi,\psi) = \left(\int_1^x \left(\frac{1}{6}x^3 - \frac{1}{2}x^2\tau\right)\psi(\tau)d\tau + \int_0^x \left(-\frac{1}{6}\tau^3 + \frac{1}{2}x\tau^2\right)\psi(\tau)d\tau + \beta^{-1}\left[\int_0^1 \tau\psi(\tau)d\tau - \alpha\phi'(0)\right]x, -\phi(x)\right),$$

$$\mathbf{B}^*\mathbf{A}^{*-1}(\phi,\psi) = (\phi(1), \ -\phi'(0)), \ \forall\,(\phi,\psi)\in\mathbf{H}.$$

Hence $\mathbf{B}^*\mathbf{A}^{*-1}$ is bounded on $\mathbf{H}$. Secondly, we consider once again the real function case only. Now, differentiate $E_m(t)$ in t to give $\dot{E}_m(t) = -\alpha\widehat{w}_{xt}^2(0,t) \le 0$. Hence $E_m(T) \le E_m(0)$ for any $T > 0$ and $\int_0^T \widehat{w}_{xt}^2(0,t)\,dt \le \alpha^{-1}E_m(0)$. Next, let $\rho_3(t) = \int_0^1 x\widehat{w}_x(x,t)\widehat{w}_t(x,t)\,dx$. Then $|\rho_3(t)| \le E_m(t)$. Since

$$\dot{\rho}_3(t) = \frac{1}{2}\widehat{w}_t^2(1,t) - \frac{1}{2}\int_0^1 [3\widehat{w}_{xx}^2(x,t) + \widehat{w}_t^2(x,t)]\,dx - (\alpha\widehat{w}_{xt}(0,t) + \beta\widehat{w}_x(0,t))\widehat{w}_x(0,t),$$

it follows that $\int_0^T \widehat{w}_t^2(1,t)\,dt \le 2(1+3T)E_m(0)$ for any $T > 0$. The result is thus proved. □

Set

$$\varepsilon(x,t) = \widehat{w}(x,t) - w(x,t). \tag{2.7}$$

Then ε satisfies

$$\begin{cases} \varepsilon_{tt}(x,t) + \varepsilon_{xxxx}(x,t) = 0, \ 0<x<1, t>0, \\ \varepsilon(0,t) = \varepsilon_{xx}(1,t) = \varepsilon_{xxx}(1,t) = 0, \\ \qquad \varepsilon_{xx}(0,t) = \alpha\varepsilon_{xt}(0,t), \ t>0. \end{cases} \tag{2.8}$$

It is well known that the above system is exponentially stable but the multiplier method is not effective in proving its exponential stability. It should be pointed out that, the error equation (2.8) explains the choice of beam configuration. Any choice of output and related observer must guarantee that the error equation (2.8) is exponentially stable. For the Euler–Bernoulli beam system, even for collocated control design, we only know that under the conservative right end boundary conditions, one of the feedback conditions as $y_{xxx}(0,t) = -\alpha y_t(0,t)$ or $y_{xxx}(0,t) = -\alpha y_{xt}(0,t)$ or $y_{xx}(0,t) = \alpha y_{xt}(0,t)$ or $y_{xx}(0,t) = \alpha y_t(0,t), \alpha > 0$ can stabilize the Euler–Bernoulli beam system. If the output does not make the error equation which takes one of these forms, we do not know whether or not our method is still effective for the stabilization of the Euler–Bernoulli beam systems.

3. Output feedback control design

Having obtained the estimated state through observer, we can now naturally design the following output feedback based on estimated state as what we have done for collocated system: $u(t) = \gamma\widehat{w}_t(1,t), \gamma > 0$. The closed-loop system now becomes

$$\begin{cases} \widehat{w}_{tt}(x,t) + \widehat{w}_{xxxx}(x,t) = 0, \ 0<x<1, t>0, \\ \widehat{w}(0,t) = \widehat{w}_{xx}(1,t) = 0, \ \widehat{w}_{xxx}(1,t) = \gamma\widehat{w}_t(1,t), \ t\ge 0 \\ \widehat{w}_{xx}(0,t) = \alpha\widehat{w}_{xt}(0,t) + \beta\widehat{w}_x(0,t) + w_{xx}(0,t), \ t\ge 0, \\ w_{tt}(x,t) + w_{xxxx}(x,t) = 0, \ 0<x<1, t>0, \\ w(0,t) = w_x(0,t) = w_{xx}(1,t) = 0, \\ \qquad w_{xxx}(1,t) = \gamma\widehat{w}_t(1,t), \ t\ge 0. \end{cases} \tag{3.1}$$

Let $\varepsilon(x,t)$ be defined by (2.7). We get the equivalent system of (3.1):

$$\begin{cases} \varepsilon_{tt}(x,t) + \varepsilon_{xxxx}(x,t) = 0, \ 0<x<1, t>0, \\ \varepsilon(0,t) = \varepsilon_{xx}(1,t) = \varepsilon_{xxx}(1,t) = 0, \\ \qquad \varepsilon_x(0,t) = \widehat{w}_x(0,t), \ t\ge 0, \\ \widehat{w}_{tt}(x,t) + \widehat{w}_{xxxx}(x,t) = 0, \ 0<x<1, t>0, \\ \widehat{w}(0,t) = \widehat{w}_{xx}(1,t) = 0, \ \widehat{w}_{xxx}(1,t) = \gamma\widehat{w}_t(1,t), \ t\ge 0 \\ \alpha\widehat{w}_{xt}(0,t) + \beta\widehat{w}_x(0,t) = \varepsilon_{xx}(0,t), \ t\ge 0. \end{cases} \tag{3.2}$$

We consider the system (3.2) in the state space $X = \{(f,g,\phi,\psi) \in \mathbf{H}\times\mathbf{H} | f'(0) = \phi'(0)\}$ with the obvious inner product induced norm: $\forall (f,g,\phi,\psi)\in X$,

$$\|(f,g,\phi,\psi)\|^2 = \int_0^1 [|f''(x)|^2 + |g(x)|^2 + |\phi''(x)|^2 + |\psi(x)|^2]\,dx + \beta|f'(0)|^2.$$

The system operator $\mathcal{A} : D(\mathcal{A})(\subset X) \to X$ for (3.2) is defined by

$$\mathcal{A}(f,g,\phi,\psi) = (g, -f^{(4)}, \psi, -\phi^{(4)}), \quad \forall\,(f,g,\phi,\psi)\in D(\mathcal{A}), \tag{3.3}$$

with $D(\mathcal{A}) = \{(f,g,\phi,\psi) \in X | \mathcal{A}(f,g,\phi,\psi) \in X, f'(0) = \phi'(0), g'(0) = \psi'(0), f''(1) = f'''(1) = \phi''(1) = 0, \alpha\psi'(0) + \beta\phi'(0) = f''(0), \phi'''(1) = \gamma\psi(1)\}$. With the operator $\mathcal{A}$ at hand, we can write (3.2) as an evolution equation in X:

$$\frac{d}{dt}(\varepsilon(\cdot,t), \varepsilon_t(\cdot,t), \widehat{w}(\cdot,t), \widehat{w}_t(\cdot,t)) = \mathcal{A}(\varepsilon(\cdot,t), \varepsilon_t(\cdot,t), \widehat{w}(\cdot,t), \widehat{w}_t(\cdot,t)). \tag{3.4}$$

We observe that $\mathcal{A}$ is not dissipative in the current inner product. Actually, let $(f,g,\phi,\psi) \in D(\mathcal{A})$. A simple computation shows that

$$\mathrm{Re}\,\langle \mathcal{A}(f,g,\phi,\psi), \ (f,g,\phi,\psi)\rangle_X = -\alpha|g'(0)|^2 - \gamma|\psi(1)|^2 - \mathrm{Re}\left(\phi''(0)\overline{\psi'(0)}\right),$$

which shows that $\mathcal{A}$ is not dissipative. Moreover, since the multiplier method is not effective in proving the stability of (2.8), it seems difficult to find an equivalent inner product in X to make $\mathcal{A}$ dissipative.

Lemma 3.1. *$\mathcal{A}^{-1}$ is compact on X and hence $\sigma(\mathcal{A})$, the spectrum of $\mathcal{A}$, consists of isolated eigenvalues only.*

Proof. For any $(p_1, q_1, p_2, q_2) \in X$, solve $\mathcal{A}(f, g, \phi, \psi) = (p_1, q_1, p_2, q_2)$ to obtain

$$\begin{cases} g(x) = p_1(x), \psi(x) = p_2(x), -f^{(4)}(x) = q_1(x), \\ \quad -\phi^{(4)}(x) = q_2(x), \\ f(0) = f''(1) = f'''(1) = \phi(0) = \phi''(1) = 0, \\ \quad f'(0) = \phi'(0), \\ \alpha\psi'(0) + \beta\phi'(0) = f''(0),\ \phi'''(1) = \gamma\psi(1). \end{cases}$$

This gives

$$\begin{cases} g(x) = p_1(x), \psi(x) = p_2(x), \phi(x) = h_2(x) \\ \quad + \gamma p_2(1)\left[\frac{1}{6}x^3 - \frac{1}{2}x^2\right] + f'(0)x, \\ f(x) = h_1(x) + f'(0)x,\ f'(0) = -\frac{1}{\beta} \\ \quad \times \left[\int_0^1 \tau q_1(\tau)\mathrm{d}\tau + \alpha p_2'(0)\right], \\ h_i(x) = \int_1^x \left[\frac{1}{2}x^2\tau - \frac{1}{6}x^3\right] q_i(\tau)\mathrm{d}\tau + \int_0^x \left[\frac{1}{6}\tau^3 - \frac{1}{2}x\tau^2\right] \\ \quad \times q_i(\tau)\mathrm{d}\tau,\ i = 1, 2. \end{cases}$$

Hence $\mathcal{A}^{-1}$ is defined everywhere on X and $\mathcal{A}^{-1}$ maps X into a subset of the space $(H^4(0,1) \times H^2(0,1))^2$, which is compact in X. By the Sobolev embedding theorem [17], $\mathcal{A}^{-1}$ is compact on X, proving the required result. □

Lemma 3.2. $\mathrm{Re}(\lambda) < 0$ *for any* $\lambda \in \sigma(\mathcal{A})$.

Proof. The proof involves only some straightforward but tedious computations without technical difficulties. We omit the details here. □

4. Asymptotic analysis for the spectrum

In this section, we study the eigenvalue problem of the closed-loop system (3.2). Let $\lambda \in \sigma(\mathcal{A})$ and $(f, g, \phi, \psi) \neq 0$ be a corresponding eigenfunction. Then $\mathcal{A}(f, g, \phi, \psi) = \lambda(f, g, \phi, \psi)$ means that $g = \lambda f$, $\psi = \lambda\phi$, and (f, ϕ) satisfies the following eigenvalue problem:

$$\begin{cases} \lambda^2 f + f^{(4)} = 0, \lambda^2\phi + \phi^{(4)} = 0, f(0) = 0, \\ \quad f''(1) = f'''(1) = \phi(0) = 0, \\ \phi''(1) = 0, f'(0) = \phi'(0), \alpha\lambda\phi'(0) + \beta\phi'(0) \\ \quad = f''(0), \phi'''(1) = \gamma\lambda\phi(1). \end{cases} \tag{4.1}$$

In this section, we study the eigenvalue problem (4.1). Using the spectral pencil theory, we first transfer (4.1) into a system of first-order ordinary differential equation parameterized by eigenvalue λ. To do this, let

$$\Phi(\cdot) = \left[f, f', f'', f''', \phi, \phi', \phi'', \phi'''\right]^\top. \tag{4.2}$$

Then (4.1) is transformed into the following equation:

$$\begin{cases} T^D(x, \lambda)\Phi(x) = \Phi'(x) + A(\lambda)\Phi(x) = 0, \\ T^R\Phi(x) = W^0(\lambda)\Phi(0) + W^1(\lambda)\Phi(1) = 0, \end{cases} \tag{4.3}$$

where $A(\lambda) = \mathrm{diag}[M(\lambda),\ M(\lambda)]$,

$$M(\lambda) = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ \lambda^2 & 0 & 0 & 0 \end{bmatrix}, W^0(\lambda) = \begin{bmatrix} W_1^0(\lambda) & 0 \\ W_2^0 & W_3^0 \\ 0 & 0 \end{bmatrix},$$

$$W^1(\lambda) = \begin{bmatrix} 0 & 0 \\ W_1^1 & 0 \\ 0 & W_2^1(\lambda) \end{bmatrix}, \tag{4.4}$$

with

$$\begin{cases} W_1^0(\lambda) := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha\lambda + \beta & -1 & 0 \end{bmatrix}, \\ W_2^0 := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \\ W_3^0 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad W_1^1 := \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \\ W_2^1(\lambda) := \begin{bmatrix} 0 & 0 & 1 & 0 \\ \gamma\lambda & 0 & 0 & -1 \end{bmatrix}. \end{cases} \tag{4.5}$$

Lemma 4.1. *Eq.* (4.1) *is equivalent to the system of the parameterized first-order ordinary differential equation* (4.3). *Moreover,* $\lambda \in \sigma(\mathcal{A})$ *if and only* (4.3) *admits a nonzero solution* Φ.

Now, let us solve problem (4.3). By Lemma 3.2 and the fact the eigenvalues are symmetric about the real again, we consider only those λ that are located in the second quadrant of the complex plane: $\lambda := \mathrm{i}\rho^2$ with $\rho \in \mathcal{S} := \left\{\rho \in \mathbb{C} \mid 0 \le \arg\rho \le \frac{\pi}{4}\right\}$. Note that for any $\rho \in \mathcal{S}$, it has $\mathrm{Re}(-\rho) \le \mathrm{Re}(\mathrm{i}\rho) \le \mathrm{Re}(-\mathrm{i}\rho) \le \mathrm{Re}(\rho)$, and

$$\begin{cases} \mathrm{Re}(-\rho) = -|\rho|\cos(\arg\rho) \le -\frac{\sqrt{2}}{2}|\rho| < 0, \\ \mathrm{Re}(\mathrm{i}\rho) = -|\rho|\sin(\arg\rho) \le 0. \end{cases}$$

For $\rho \in \mathbb{C}$, $\rho \neq 0$, define an invertible (diagonal block) matrix function $P(\rho) = \mathrm{diag}[P_1(\rho),\ P_1(\rho)]$ with

$$P_1(\rho) := \begin{bmatrix} \rho & \rho & \rho & \rho \\ \rho^2 & -\rho^2 & \mathrm{i}\rho^2 & -\mathrm{i}\rho^2 \\ \rho^3 & \rho^3 & -\rho^3 & -\rho^3 \\ \rho^4 & -\rho^4 & -\mathrm{i}\rho^4 & \mathrm{i}\rho^4 \end{bmatrix},$$

$$P_1^{-1}(\rho) = \begin{bmatrix} \frac{1}{4\rho} & \frac{1}{4\rho^2} & \frac{1}{4\rho^3} & \frac{1}{4\rho^4} \\ \frac{1}{4\rho} & -\frac{1}{4\rho^2} & \frac{1}{4\rho^3} & -\frac{1}{4\rho^4} \\ \frac{1}{4\rho} & -\mathrm{i}\frac{1}{4\rho^2} & -\frac{1}{4\rho^3} & \mathrm{i}\frac{1}{4\rho^4} \\ \frac{1}{4\rho} & \mathrm{i}\frac{1}{4\rho^2} & -\frac{1}{4\rho^3} & -\mathrm{i}\frac{1}{4\rho^4} \end{bmatrix}. \tag{4.6}$$

Using P^{-1}, we can make an invertible linear transformation to (4.3) with $\lambda = \mathrm{i}\rho^2$:

$$\Psi(x) := P^{-1}(\rho)\Phi(x),$$
$$\widehat{T}^D(x, \rho) := P(\rho)^{-1}T^D(x, \mathrm{i}\rho^2)P(\rho). \tag{4.7}$$

It then has

$$\widehat{T}^D(x,\rho)\Psi(x) = \Psi'(x) + \widehat{A}(\rho)\Psi(x) = 0, \tag{4.8}$$

where $\widehat{A}(\rho) = P(\rho)^{-1}A(\mathrm{i}\rho^2)P(\rho) = \mathrm{diag}[\widehat{A}_1(\rho), \ \widehat{A}_1(\rho)]$, with

$$\begin{aligned}\widehat{A}_1(\rho) &:= P_1^{-1}(\rho)M(\mathrm{i}\rho^2)P_1(\rho)\\ &= \mathrm{diag}\,[-\rho,\ \rho,\ -\mathrm{i}\rho,\ \mathrm{i}\rho].\end{aligned} \tag{4.9}$$

Lemma 4.2. *Let $0 \neq \rho \in \mathcal{S}$. For $x \in [0,1]$, there exists a fundamental matrix solution to problem (4.8) given by $E(x,\rho) = \mathrm{diag}[E_1(x,\rho),\ E_1(x,\rho)]$, where*

$$E_1(x,\rho) := \mathrm{diag}\left[\mathrm{e}^{\rho x},\ \mathrm{e}^{-\rho x},\ \mathrm{e}^{\mathrm{i}\rho x},\ \mathrm{e}^{-\mathrm{i}\rho x}\right]. \tag{4.10}$$

By (4.7),

$$\widehat{\Phi}(x,\rho) := P(\rho)E(x,\rho) \tag{4.11}$$

is a fundamental matrix solution to (4.3) with $\lambda = \mathrm{i}\rho^2$, $\rho \in \mathcal{S}$.

Next, we are ready to estimate asymptotically the distribution of eigenvalues of $\mathcal{A}$ on the complex plane. From (4.3), $\lambda = \mathrm{i}\rho^2 \in \sigma(\mathcal{A})$ with $\rho \in \mathcal{S}$ if and only if ρ is a zero of the characteristic determinant $\Delta(\rho)$:

$$\Delta(\rho) := \det\left(T^R(\mathrm{i}\rho^2)\widehat{\Phi}\right), \quad \rho \in \mathcal{S}, \tag{4.12}$$

where T^R is defined in (4.3) and $\widehat{\Phi}$ is given by (4.11).

Theorem 4.1. *Let $\Delta(\rho)$ be the characteristic determinant of the system (4.3) in the sector $\mathcal{S}$ with $\lambda = \mathrm{i}\rho^2$ defined by (4.12). Then $\Delta(\rho) = \Delta_1(\rho)\Delta_2(\rho)$, for $\Delta_1(\rho)$ and $\Delta_2(\rho)$ see Box I.*

$\Delta_i, i = 1,2$ have the following asymptotic expressions, respectively:

$$\begin{aligned}\Delta_1(\rho) = -2\rho^{12}\mathrm{e}^{\rho}\bigg\{&\left(\alpha - \frac{\mathrm{i}+1}{\rho}\right)\mathrm{e}^{-\mathrm{i}\rho}\\ &+\left(\alpha - \frac{\mathrm{i}-1}{\rho}\right)\mathrm{e}^{\mathrm{i}\rho} + \mathcal{O}\left(\frac{1}{\rho^2}\right)\bigg\},\end{aligned} \tag{4.13}$$

$$\begin{aligned}\Delta_2(\rho) = -2\mathrm{i}\rho^{10}\mathrm{e}^{\rho}\bigg\{&\left(1 - \frac{(i+1)\gamma}{\rho}\right)\mathrm{e}^{-\mathrm{i}\rho}\\ &+\left(1 - \frac{(i-1)\gamma}{\rho}\right)\mathrm{e}^{\mathrm{i}\rho} + \mathcal{O}\left(\frac{1}{\rho^2}\right)\bigg\}.\end{aligned} \tag{4.14}$$

Furthermore, the eigenvalues $\{\lambda_{1n}, \overline{\lambda}_{1n}, \lambda_{2n}, \overline{\lambda}_{2n},\ n \in \mathbb{N}\}$ of the system (4.3) have the following asymptotic expansions: as $n \to \infty$,

$$\begin{aligned}\lambda_{1n} &= -\frac{2}{\alpha} + \mathrm{i}\left(\frac{1}{2}+n\right)^2\pi^2 + \mathcal{O}\left(\frac{1}{n}\right),\\ \lambda_{2n} &= -2\gamma + \mathrm{i}\left(\frac{1}{2}+n\right)^2\pi^2 + \mathcal{O}\left(\frac{1}{n}\right),\end{aligned} \tag{4.15}$$

where n's are positive integers. Therefore as $n \to \infty$,

$$\mathrm{Re}\{\lambda_{1n}, \overline{\lambda}_{1n}\} \to -\frac{2}{\alpha}, \quad \mathrm{Re}\{\lambda_{2n}, \overline{\lambda}_{2n}\} \to -2\gamma. \tag{4.16}$$

Proof. Since $T^R(\mathrm{i}\rho^2)\widehat{\Phi} = W^0(\mathrm{i}\rho^2)P(\rho)E(0,\rho) + W^1(\mathrm{i}\rho^2)P(\rho)E(1,\rho)$, it follows from (4.4) and (4.5) that

$$T^R(\mathrm{i}\rho^2)\widehat{\Phi} = \begin{bmatrix}\widehat{W}_1^0(\rho) & 0\\ \widehat{W}_2^0(\rho) & \widehat{W}_3^0(\rho)\\ \widehat{W}_1^1(\rho) & 0\\ 0 & \widehat{W}_2^1(\rho)\end{bmatrix},$$

$$\Delta(\rho) = \det\left(T^R(\mathrm{i}\rho^2)\widehat{\Phi}\right), \tag{4.17}$$

where $\widehat{W}_1^0(\rho)$, $\widehat{W}_2^0(\rho)$, $\widehat{W}_3^0(\rho)$, $\widehat{W}_1^1(\rho)$ and $\widehat{W}_2^1(\rho)$ are given in Box II.

Hence, $\Delta(\rho) = \Delta_1(\rho)\Delta_2(\rho)$, where

$$\Delta_1(\rho) = \det\begin{bmatrix}\widehat{W}_1^0(\rho)\\ \widehat{W}_1^1(\rho)\end{bmatrix}, \qquad \Delta_2(\rho) = \det\begin{bmatrix}\widehat{W}_3^0(\rho)\\ \widehat{W}_2^1(\rho)\end{bmatrix}. \tag{4.18}$$

Further direct computations give (4.13) and (4.14), respectively.

Now set $\rho \in \mathcal{S}$. Then $\Delta(\rho) = 0$ if and only if either $\Delta_1(\rho) = 0$ or $\Delta_2(\rho) = 0$. If $\Delta_1(\rho) = 0$, by (4.13), it has

$$\left(1 - \frac{\mathrm{i}+1}{\alpha\rho}\right)\mathrm{e}^{-\mathrm{i}\rho} + \left(1 - \frac{\mathrm{i}-1}{\alpha\rho}\right)\mathrm{e}^{\mathrm{i}\rho} + \mathcal{O}(\rho^{-2}) = 0. \tag{4.19}$$

This leads to

$$\mathrm{e}^{-\mathrm{i}\rho} + \mathrm{e}^{\mathrm{i}\rho} + \mathcal{O}(\rho^{-1}) = 0. \tag{4.20}$$

Note that in the first quadrant of the complex plane, the solutions of the equation $\mathrm{e}^{\mathrm{i}\rho} + \mathrm{e}^{-\mathrm{i}\rho} = 0$ are given by $\tilde{\rho}_{1n} = \left(\frac{1}{2}+n\right)\pi$, $n = 0,1,2,\ldots$. Apply Rouché's theorem to (4.20) to give the solutions of (4.20):

$$\rho_{1n} = \tilde{\rho}_{1n} + \alpha_{1n} = \left(\frac{1}{2}+n\right)\pi + \alpha_{1n},\ \alpha_{1n} = \mathcal{O}(n^{-1}) \quad \text{as } n \to \infty. \tag{4.21}$$

Substitute ρ_{1n} into (4.19) and use the fact $\mathrm{e}^{\mathrm{i}\tilde{\rho}_{1n}} = -\mathrm{e}^{-\mathrm{i}\tilde{\rho}_{1n}}$, to obtain

$$\left(1 - \frac{\mathrm{i}-1}{\alpha\rho_{1n}}\right)\mathrm{e}^{\mathrm{i}\alpha_{1n}} - \left(1 - \frac{\mathrm{i}+1}{\alpha\rho_{1n}}\right)\mathrm{e}^{-\mathrm{i}\alpha_{1n}} + \mathcal{O}(\rho_{1n}^{-2}) = 0.$$

Expand the exponential functions above into Taylor series to yield $\alpha_{1n} = -(\mathrm{i}\alpha(\frac{1}{2}+n)\pi)^{-1} + \mathcal{O}(n^{-2})$. Substitute this into (4.21) to produce

$$\rho_{1n} = \left(\frac{1}{2}+n\right)\pi - \frac{1}{\mathrm{i}\alpha\left(\frac{1}{2}+n\right)\pi} + \mathcal{O}(n^{-2}) \quad \text{as } n \to \infty. \tag{4.22}$$

Since $\lambda_{1n} = \mathrm{i}\rho_{1n}^2$, we get eventually its asymptotic expression given by (4.15). Similar analysis gives another branch of eigenvalues $\lambda_{2n} = \mathrm{i}\rho_{2n}^2$ in (4.15), where

$$\rho_{2n} = \left(\frac{1}{2}+n\right)\pi - \frac{k}{\mathrm{i}\left(\frac{1}{2}+n\right)\pi} + \mathcal{O}(n^{-2}) \quad \text{as } n \to \infty. \quad \square \tag{4.23}$$

$$\Delta_1(\rho)=\rho^{12}\det\begin{bmatrix}1 & 1 & 1 & 1\\ \mathrm{i}\alpha-\frac{1}{\rho}+\frac{\beta}{\rho^2} & -\mathrm{i}\alpha-\frac{1}{\rho}-\frac{\beta}{\rho^2} & -\alpha+\frac{1}{\rho}+\frac{\mathrm{i}\beta}{\rho^2} & \alpha+\frac{1}{\rho}-\frac{\mathrm{i}\beta}{\rho^2}\\ \mathrm{e}^{\rho} & \mathrm{e}^{-\rho} & -\mathrm{e}^{\mathrm{i}\rho} & -\mathrm{e}^{-\mathrm{i}\rho}\\ \mathrm{e}^{\rho} & -\mathrm{e}^{-\rho} & -\mathrm{i}\mathrm{e}^{\mathrm{i}\rho} & \mathrm{i}\mathrm{e}^{-\mathrm{i}\rho}\end{bmatrix}$$

$$\Delta_2(\rho)=\rho^{10}\det\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & -1 & \mathrm{i} & -i\\ \mathrm{e}^{\rho} & \mathrm{e}^{-\rho} & -\mathrm{e}^{\mathrm{i}\rho} & -\mathrm{e}^{-\mathrm{i}\rho}\\ \left(-1+\frac{\mathrm{i}\gamma}{\rho}\right)\mathrm{e}^{\rho} & \left(1+\frac{\mathrm{i}\gamma}{\rho}\right)\mathrm{e}^{-\rho} & \left(i+\frac{\mathrm{i}\gamma}{\rho}\right)\mathrm{e}^{\mathrm{i}\rho} & \left(-i+\frac{\mathrm{i}\gamma}{\rho}\right)\mathrm{e}^{-\mathrm{i}\rho}\end{bmatrix}$$

Box I.

$$\widehat{W}_1^0(\rho)=\rho^4\begin{bmatrix}\rho^{-3} & \rho^{-3} & \rho^{-3} & \rho^{-3}\\ \mathrm{i}\alpha-\frac{1}{\rho}+\frac{\beta}{\rho^2} & -\mathrm{i}\alpha-\frac{1}{\rho}-\frac{\beta}{\rho^2} & -\alpha+\frac{1}{\rho}+\frac{\mathrm{i}\beta}{\rho^2} & \alpha+\frac{1}{\rho}-\frac{\mathrm{i}\beta}{\rho^2}\end{bmatrix}$$

$$\widehat{W}_2^0(\rho)=\begin{bmatrix}0 & 0 & 0 & 0\\ -\rho^2 & \rho^2 & -\mathrm{i}\rho^2 & \mathrm{i}\rho^2\end{bmatrix},\ \widehat{W}_3^0(\rho)=\begin{bmatrix}\rho & \rho & \rho & \rho\\ \rho^2 & -\rho^2 & \mathrm{i}\rho^2 & -\mathrm{i}\rho^2\end{bmatrix}$$

$$\widehat{W}_1^1(\rho)=\begin{bmatrix}\rho^3\mathrm{e}^{\rho} & \rho^3\mathrm{e}^{-\rho} & -\rho^3\mathrm{e}^{\mathrm{i}\rho} & -\rho^3\mathrm{e}^{-\mathrm{i}\rho}\\ \rho^4\mathrm{e}^{\rho} & -\rho^4\mathrm{e}^{-\rho} & -\mathrm{i}\rho^4\mathrm{e}^{\mathrm{i}\rho} & \mathrm{i}\rho^4\mathrm{e}^{-\mathrm{i}\rho}\end{bmatrix}$$

$$\widehat{W}_2^1(\rho)=\begin{bmatrix}\rho^3\mathrm{e}^{\rho} & \rho^3\mathrm{e}^{-\rho} & -\rho^3\mathrm{e}^{\mathrm{i}\rho} & -\rho^3\mathrm{e}^{-\mathrm{i}\rho}\\ \rho^4\left(-1+\frac{\mathrm{i}\gamma}{\rho}\right)\mathrm{e}^{\rho} & \rho^4\left(1+\frac{\mathrm{i}\gamma}{\rho}\right)\mathrm{e}^{-\rho} & \rho^4\left(\mathrm{i}+\frac{\mathrm{i}\gamma}{\rho}\right)\mathrm{e}^{\mathrm{i}\rho} & \rho^4\left(-\mathrm{i}+\frac{\mathrm{i}\gamma}{\rho}\right)\mathrm{e}^{-\mathrm{i}\rho}\end{bmatrix}$$

Box II.

Theorem 4.2. *Let $\{\lambda_{1n},\overline{\lambda}_{1n},\lambda_{2n},\overline{\lambda}_{2n},\quad n\in\mathbb{N}\}$ be the eigenvalues of $\mathcal{A}$ with λ_{in} being given in (4.15). Then the corresponding eigenfunctions*

$$\left\{[f''_{in}(x),\lambda_{in}f_{in}(x),\phi''_{in}(x),\lambda_{in}\phi_{in}(x)],\quad i=1,2,n\in\mathbb{N}\right\}\tag{4.24}$$

have the following asymptotic expressions:

$$\begin{cases} f''_{1n}(x)=(\mathrm{i}+1)\mathrm{e}^{-\mathrm{i}\rho_{1n}(1-x)}-2\mathrm{i}\mathrm{e}^{-\rho_{1n}(1-x)}+2\mathrm{e}^{-\mathrm{i}\rho_{1n}}\mathrm{e}^{-\rho_{1n}x}\\ \qquad -(1-\mathrm{i})\mathrm{e}^{\mathrm{i}\rho_{1n}(1-x)}+\mathcal{O}(n^{-1}),\\ \lambda_{1n}f_{1n}(x)=2\mathrm{e}^{-\rho_{1n}(1-x)}+2\mathrm{i}\mathrm{e}^{-\mathrm{i}\rho_{1n}}\mathrm{e}^{-\rho_{1n}x}\\ \quad +(1-\mathrm{i})\mathrm{e}^{-\mathrm{i}\rho_{1n}(1-x)}+(1+\mathrm{i})\mathrm{e}^{\mathrm{i}\rho_{1n}(1-x)}+\mathcal{O}(n^{-1}),\\ f'_{1n}(x)=\mathcal{O}(n^{-1}),\ \phi''_{1n}(x)=f''_{1n}(x)+\mathcal{O}(n^{-1}),\\ \lambda_{1n}\phi_{1n}(x)=\lambda_{1n}f_{1n}(x)+\mathcal{O}(n^{-1}),\\ f''_{2n}(x)=\lambda_{2n}f_{2n}(x)=0,\\ \phi''_{2n}(x)=(1-\mathrm{i})\mathrm{e}^{\mathrm{i}\rho_{2n}x}-(1+\mathrm{i})\mathrm{e}^{-\mathrm{i}\rho_{2n}x}\\ \quad +2\mathrm{e}^{-\mathrm{i}\rho_{2n}}\mathrm{e}^{-\rho_{2n}(1-x)}-2\mathrm{i}\mathrm{e}^{-\rho_{2n}x}+\mathcal{O}(n^{-1}),\\ \lambda_{2n}\phi_{2n}(x)=(\mathrm{i}-1)\mathrm{e}^{-\mathrm{i}\rho_{2n}x}+2\mathrm{i}\mathrm{e}^{-\mathrm{i}\rho_{2n}}\mathrm{e}^{-\rho_{2n}(1-x)}\\ \qquad -(1+\mathrm{i})\mathrm{e}^{\mathrm{i}\rho_{2n}x}+2\mathrm{e}^{-\rho_{2n}x}+\mathcal{O}(n^{-1}), \end{cases}\tag{4.25}$$

where ρ_{1n} and ρ_{2n} are given by (4.22) and (4.23) respectively. Moreover,

$$\{[f_{in},\ \lambda_{in}f_{in},\ \phi_{in},\ \lambda_{in}\phi_{in}],\ i=1,2;\ n\in\mathbb{N}\}$$

are approximately normalized in X in the sense that there exist positive constants c_1 and c_2, independent of n, such that for $n\in\mathbb{N}$,

$$c_1\le\|f''_{1n}\|_{L^2}+|f'_{1n}(0)|,\ \|f''_{2n}\|_{L^2},\ \|\lambda_{in}f_{in}\|_{L^2},\\ \|\phi''_{in}\|_{L^2},\ \|\lambda_{in}\phi_{in}\|_{L^2}\le c_2.\tag{4.26}$$

Proof. A nonzero solution $\Phi(x)$ of (4.2) associated with eigenvalue λ can be obtained in this way: its jth component is the determinant of the matrix that is the $T^R\widehat{\Phi}$ in (4.17) but replace one of the rows of $T^R\widehat{\Phi}$ with $e_j^\top(\widehat{\Phi}(x,\lambda))$ so that such obtained $\Phi(x)$ is not identical to zero, where e_j is the jth column of the identity matrix.

For λ_{1n}, each component of corresponding $\Phi_{1n}(x)$ can be determined by replacing the second row of $T^R\widehat{\Phi}$ in (4.17) with $e_j^\top(\widehat{\Phi}(x,\lambda))$.

Thus the first component of $\Phi_1(x)$ is given by (for brief of notation, we omit the subscript without confusion)

$$f_1(x,\rho)=\rho^9\mathrm{e}^{\rho}\Delta_2(\rho)\times\{-2\mathrm{i}\mathrm{e}^{-\rho(1-x)}\\ -(\mathrm{i}+1)\mathrm{e}^{-\mathrm{i}\rho(1-x)}+(\mathrm{i}+1)\mathrm{e}^{-\mathrm{i}\rho}\mathrm{e}^{-\rho x}\\ +(1-\mathrm{i})\mathrm{e}^{\mathrm{i}\rho(1-x)}-(1-\mathrm{i})\mathrm{e}^{\mathrm{i}\rho}\mathrm{e}^{-\rho x}+\mathcal{O}(\mathrm{e}^{-c|\rho|})\},$$

where c is an arbitrary small positive constant. By (4.20), it has

$$f_1(x,\rho)=\rho^9\mathrm{e}^{\rho}\Delta_2(\rho)\times\{2\mathrm{e}^{-\mathrm{i}\rho}\mathrm{e}^{-\rho x}-2\mathrm{i}\mathrm{e}^{-\rho(1-x)}\\ -(\mathrm{i}+1)\mathrm{e}^{-\mathrm{i}\rho(1-x)}+(1-\mathrm{i})\mathrm{e}^{\mathrm{i}\rho(1-x)}+\mathcal{O}(\rho^{-1})\}.\tag{4.27}$$

Similarly,

$$\begin{cases} f_1'(x,\rho) = \rho^{10}e^{\rho}\Delta_2(\rho)\times\{-2e^{-i\rho}e^{-\rho x} - 2ie^{-\rho(1-x)} \\ \quad +(1-i)e^{-i\rho(1-x)} - (1+i)e^{i\rho(1-x)} + \mathcal{O}(\rho^{-1})\}, \\ f_1''(x,\rho) = \rho^{11}e^{\rho}\Delta_2(\rho)\times\{2e^{-i\rho}e^{-\rho x} \\ \quad -2ie^{-\rho(1-x)} + (i+1)e^{-i\rho(1-x)} \\ \quad -(1-i)e^{i\rho(1-x)} + \mathcal{O}(\rho^{-1})\}, \\ \phi_1(x,\rho) = \rho^{9}e^{\rho}\Delta_3(\rho)\times\{2e^{-i\rho}e^{-\rho x} - 2ie^{-\rho(1-x)} \\ \quad -(i+1)e^{-i\rho(1-x)} + (1-i)e^{i\rho(1-x)} + \mathcal{O}(\rho^{-1})\}, \\ \phi_1''(x,\rho) = \rho^{11}e^{\rho}\Delta_3(\rho)\times\{2e^{-i\rho}e^{-\rho x} - 2ie^{-\rho(1-x)} \\ \quad +(i+1)e^{-i\rho(1-x)} - (1-i)e^{i\rho(1-x)} + \mathcal{O}(\rho^{-1})\}, \end{cases} \tag{4.28}$$

where

$$\Delta_3(\rho) := \det\begin{bmatrix} \rho & \rho & \rho & \rho \\ -\rho^2 & \rho^2 & -i\rho^2 & i\rho^2 \\ \rho^3e^{\rho} & \rho^3e^{-\rho} & -\rho^3e^{i\rho} & -\rho^3e^{-i\rho} \\ \rho^4e^{\rho} & -\rho^4e^{-\rho} & -i\rho^4e^{i\rho} & i\rho^4e^{-i\rho} \end{bmatrix}$$

$$= \Delta_2(\rho) + \mathcal{O}(\rho^{-1}). \tag{4.29}$$

For λ_{2n}, each component of $\Phi_{2n}(x)$ can be determined in the same way stated in the beginning of the proof by replacing the last (eighth) row of $T^R\widehat{\Phi}$ in (4.17) with $e_j^\top(\widehat{\Phi}(x,\lambda))$.

In this way, we have (again for brief of notation, we omit the subscript without confusion)

$$f_2(x,\rho) = f_2''(x,\rho) = 0, \tag{4.30}$$

$$\phi_2(x,\rho) = \rho^7e^{\rho}\Delta_1(\rho)\times\{(1+i)e^{-i\rho x} + 2e^{-i\rho}e^{-\rho(1-x)} \\ -(1-i)e^{i\rho x} - 2ie^{-\rho x} + \mathcal{O}(\rho^{-1})\}, \tag{4.31}$$

$$\phi_2''(x,\rho) = \rho^9e^{\rho}\Delta_1(\rho)\times\{(1-i)e^{i\rho x} - (1+i)e^{-i\rho x} \\ +2e^{-i\rho}e^{-\rho(1-x)} - 2ie^{-\rho x} + \mathcal{O}(\rho^{-1})\}. \tag{4.32}$$

Notice that $\Delta_1(\rho_{2n}) \neq 0$, $\Delta_2(\rho_{1n}) \neq 0$ and $\Delta_3(\rho_{2n}) \neq 0$, where Δ_3 is defined by (4.29) and ρ_{1n}, ρ_{2n} are given by (4.22) and (4.23) respectively. Then (4.25) can be deduced from (4.27)–(4.32) by setting $f_{2n}(x) = 0$,

$$f_{1n}(x) = \frac{f_1(x,\rho_{1n})}{\rho^{11}e^{\rho}\Delta_2(\rho_{1n})}, \quad \phi_{1n}(x) = \frac{\phi_1(x,\rho_{1n})}{\rho^{11}e^{\rho}\Delta_2(\rho_{1n})},$$

$$\phi_{2n}(x) = \frac{\phi_2(x,\rho_{2n})}{\rho^9e^{\rho}\Delta_1(\rho_{2n})}.$$

Finally, it follows from (4.22) and (4.23) that

$$\|e^{-\rho_{in}x}\|_{L^2} = \mathcal{O}(n^{-1}), \quad \|e^{-i\rho_{in}x}\|_{L^2} = 1 + \mathcal{O}(n^{-1}) \quad \text{for } i = 1, 2.$$

These together with (4.25) yield (4.26). The proof is complete. □

5. Riesz basis generation and exponential stability

This section is devoted to the Riesz basis property for system (3.2). The main result is the following Theorem 5.1.

Theorem 5.1. *Let $\mathcal{A}$ be defined by (3.3). Then each eigenvalue with large modulus is algebraically simple. Moreover, there is a set of generalized eigenfunctions of $\mathcal{A}$, which forms a Riesz basis for X.*

Proof. Let $\mathbf{A}$ be given in $\mathbf{H}$ by (2.2) and let $\lambda = i\rho^2$. It is easy to check that $\Delta_1(\rho)$ given in Box I is also the characteristic determinant of $\mathbf{A}$, which has the asymptotic expression (4.13). The eigenvalues of $\mathbf{A}$, which is denoted by $\{\lambda_{1n}, \overline{\lambda}_{1n}, n \in \mathbb{N}\}$ and corresponding generalized eigenfunctions $\{(f_{1n}, \lambda_{1n}f_{1n}), \overline{(f_{1n}, \lambda_{1n}f_{1n})}, n \in \mathbb{N}\}$ have the asymptotic expressions (4.15) and (4.25), respectively. In terms of regular theory of the second-order partial differential equations (see e.g., [11]), we know that each eigenvalue is algebraically simple when its modulus is large enough and there is a set of generalized eigenfunctions of $\mathbf{A}$, which forms a Riesz basis for $\mathbf{H}$. Now, there is an isometric isomorphism $\mathbb{T}_1$ between $\mathbf{H}$ and $L^2(0,1)\times L^2(0,1)\times\mathbb{C}^1$ given in

$$\mathbb{T}_1(f,g) = (f'', g, \beta f'(0)), \quad \forall (f,g) \in \mathbf{H}.$$

Therefore, $\{(f_{1n}'', \lambda_{1n}f_{1n}, \beta f_{1n}'(0)), n \in \mathbb{N}\} \cup \{(\overline{f_{1n}''}, \overline{\lambda_{1n}}\ \overline{f_{1n}}, \beta\overline{f_{1n}'}(0)), n \in \mathbb{N}\}$ forms a Riesz basis for $L^2(0,1)\times L^2(0,1)\times\mathbb{C}^1$.

Next, let $\mathbb{A}$ be the operator defined in $\mathbb{H}$ by (1.3) and again let $\lambda = i\rho^2$. We have

(a) $\Delta_2(\rho)$ given by Box I is the characteristic determinant of eigenvalues of $\mathbb{A}$, which has the asymptotic expression (4.14);
(b) the eigenvalues $\{\lambda_{2n}, \overline{\lambda_{2n}}, n \in \mathbb{N}\}$ have the asymptotic expansions (4.15) and each eigenvalue is algebraically simple when its modulus is large enough;
(c) the corresponding eigenfunctions $\{(\phi_{2n}, \lambda_{2n}\phi_{2n}), \overline{(\phi_{2n}, \lambda_{2n}\phi_{2n})}, n \in \mathbb{N}\}$ have the asymptotic expressions (4.25);
(d) there is a set of generalized eigenfunctions of $\mathbb{A}$, which forms a Riesz basis for $\mathbb{H}$.

Define an isometric isomorphism $\mathbb{T}_2 : \mathbb{H} \to L^2(0,1)\times L^2(0,1)$ by

$$\mathbb{T}_2(\phi,\psi) = (\phi'', \psi), \ \forall (\phi,\psi) \in \mathbb{H}.$$

Then $\{(\phi_{2n}'', \lambda_{2n}\phi_{2n}), n \in \mathbb{N}\} \cup \{(\overline{\phi_{2n}''}, \overline{\lambda_{2n}\phi_{2n}}), n \in \mathbb{N}\}$ form a Riesz basis for $L^2(0,1)\times L^2(0,1)$.

To sum up, we have obtained that $\{(f_{1n}'', \lambda_{1n}f_{1n}, \beta f_{1n}'(0), 0, 0), (0,0,0,\phi_{2n}'', \lambda_{2n}\phi_{2n}), n \in \mathbb{N}\}$ and their conjugates $(\{\overline{f_{1n}''}, \overline{\lambda_{1n}f_{1n}}, \overline{\beta f_{1n}'(0)}, 0, 0), (0,0,0,\overline{\phi_{2n}''}, \overline{\lambda_{2n}\phi_{2n}}), n \in \mathbb{N}\}$ form a Riesz basis for $(L^2(0,1))^2\times\mathbb{C}^1\times(L^2(0,1))^2$. Finally, define an isometric isomorphism $\mathbb{T}_3 : X \to (L^2(0,1))^2\times\mathbb{C}^1\times(L^2(0,1))^2$ by

$$\mathbb{T}_2(f,g,\phi,\psi) = (f'', g, \beta f'(0), \phi'', \psi), \quad \forall (f,g,\phi,\psi) \in X.$$

Then $\{(f_{1n}, \lambda_{1n}f_{1n}, 0, 0), (0, 0, \phi_{2n}, \lambda_{2n}\phi_{2n}), n \in \mathbb{N}\} \cup \{(\overline{f_{1n}}, \overline{\lambda_{1n}f_{1n}}, 0, 0), (0, 0, \overline{\phi_{2n}}, \overline{\lambda_{2n}\phi_{2n}}), n \in \mathbb{N}\}$ also forms a Riesz basis for X. Since

$$\begin{pmatrix} \mathbf{I}_{2\times2} & \mathbf{0}_{2\times2} \\ \mathbf{I}_{2\times2} & \mathbf{I}_{2\times2} \end{pmatrix}(f_{1n}, \lambda_{1n}f_{1n}, 0, 0)^{\mathrm{T}} = (f_{1n}, \lambda_{1n}f_{1n}, f_{1n}, \lambda_{1n}f_{1n})^{\mathrm{T}}$$

and

$$\begin{pmatrix} \mathbf{I}_{2\times2} & \mathbf{0}_{2\times2} \\ \mathbf{I}_{2\times2} & \mathbf{I}_{2\times2} \end{pmatrix}(0, 0, \phi_{2n}, \lambda_{2n}\phi_{2n})^{\mathrm{T}} = (0, 0, \phi_{2n}, \lambda_{2n}\phi_{2n})^{\mathrm{T}},$$

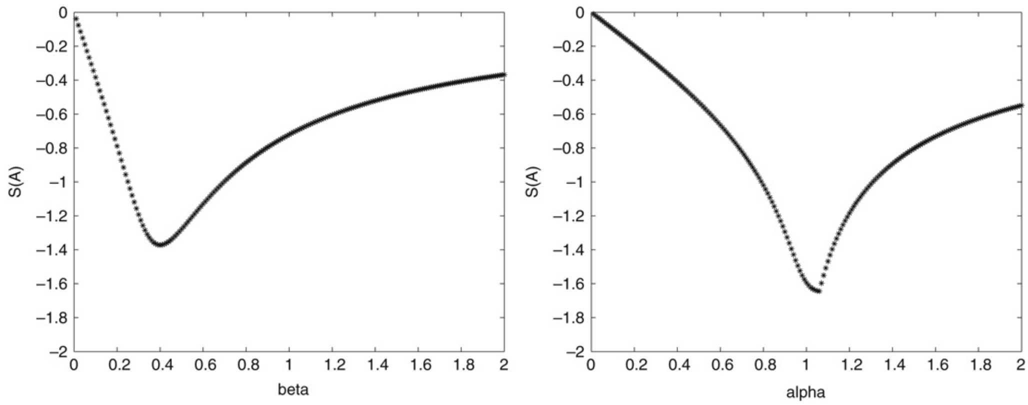

Fig. 1. The functional relation of $S(\mathcal{A})$ with respect to β with $\alpha = 1, \gamma = 1$ (left) and α with $\beta = 1, \gamma = 1$ (right).

we conclude that $\{(f_{1n}, \lambda_{1n} f_{1n}, f_{1n}, \lambda_{1n} f_{1n})(0, 0, \phi_{2n},, \lambda_{2n}\phi_{2n}), n \in \mathbb{N}\}$ together with their conjugates $\{(\overline{f_{1n}}, \overline{\lambda_{1n} f_{1n}}, \overline{f_{1n}}, \overline{\lambda_{1n} f_{1n}}), (0, 0, \overline{\phi_{2n}}, \overline{\lambda_{2n}\phi_{2n}}), n \in \mathbb{N}\}$ form a Riesz basis for X. Now, by a variant Bari's theorem (see [10]) and the expressions (4.25), we know that each eigenvalue is algebraically simple when its modulus is large enough and there is a set of generalized eigenfunctions of $\mathcal{A}$, which forms a Riesz basis for X. The proof is complete. □

Theorem 5.2. *Let $\mathcal{A}$ be defined by (3.3). Then*

(i) *$\mathcal{A}$ generates a C_0-semigroup $e^{\mathcal{A}t}$ on X.*
(ii) *The spectrum-determined growth condition holds true for $e^{\mathcal{A}t}$, that is to say, $S(\mathcal{A}) = \omega(\mathcal{A})$, where $S(\mathcal{A}) := \sup_{\lambda\in\sigma(\mathcal{A})} \operatorname{Re}\lambda$ is the spectral bound of $\mathcal{A}$, and $\omega(\mathcal{A}) := \inf\{\omega | \exists M > 0$ such that $\|e^{\mathcal{A}t}\| \le Me^{\omega t}\}$ is the growth order of $e^{\mathcal{A}t}$.*
(iii) *System (3.2) is exponentially stable.*

Proof. Since there is a set of generalized eigenfunctions of $\mathcal{A}$, which forms a Riesz basis for X, (i) and (ii) then follow from the asymptotic expansion (4.15) for eigenvalues. By (ii), the stability of (3.2) can be determined by the maximal value of the real parts of eigenvalues of $\mathcal{A}$. Now, by Lemma 3.2, $\operatorname{Re}(\lambda) < 0$ for any $\lambda \in \sigma(\mathcal{A})$, and from Theorem 4.1, the imaginary axis is not the asymptote of the eigenvalues. Hence, system (3.2) is exponentially stable. The proof is complete. □

6. Concluding remarks

In this paper, we design, for a non-collocated Euler–Bernoulli beam, an infinite-dimensional observer. The admissibility of the observation of the original system and solvability of the observer are discussed by well-posed linear infinite-dimensional system theory that has been well developed in the last two decades. The estimated output feedback is designed and the stability of the closed-loop system is discussed by Riesz basis approach. The advantage of the Riesz basis approach is that the choice of beam configuration is not crucial for the approach but crucial for Lyapunov function and the multiplier methods that are commonly used in the stability analysis for PDEs. Actually, for the system we are concerned with, the multiplier method is not effective in proving the stability of the closed-loop system.

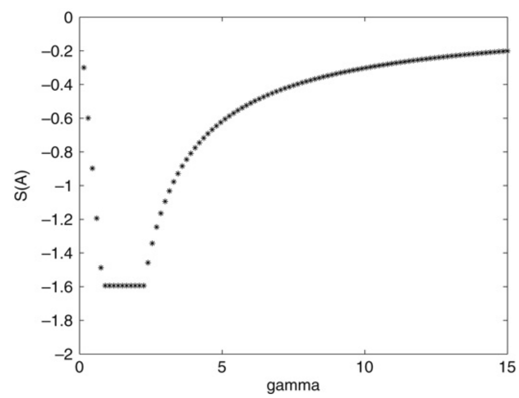

Fig. 2. The functional relation of $S(\mathcal{A})$ with respect to γ with $\alpha = 1, \beta = 1$.

Moreover, the decay rate for the non-collocated system is not proportional to the feedback gain. There exists an optimal range for the feedback gain. Both larger and smaller feedback gains would make the low eigenfrequencies approach the imaginary axis. This is in contrast to the high eigenfrequencies (4.16). The numerical simulations of Figs. 1 and 2 confirm these facts: the decay rate $S(\mathcal{A}) = \omega(\mathcal{A})$ would increase when the gains α, β, γ increase to infinity or decrease to zero.

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