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Riesz basis and stabilization for the flexible structure of a symmetric tree-shaped beam network

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SUMMARY

The stabilization of a symmetric tree-shaped network of Euler–Bernoulli beams described by a system of partial differential equations is considered. The boundary controllers are designed based on passivity principle. The eigenfrequencies are analysed in detail and the asymptotic expansion of eigenvalues are presented. It is shown that there is a set of generalized eigenfunctions for the closed-loop system, which forms a Riesz basis with parentheses for the energy state space. This concludes the spectrum-determined growth condition and the exponential stability of the closed-loop system. Copyright © 2007 John Wiley & Sons, Ltd.

KEY WORDS: beam network; collocated control; spectral analysis; exponential stability

1. INTRODUCTION

In engineering and sciences, due to initial configurations of materials or weight limitations, flexible structures consisting of finitely many connected flexible elements such as strings, beams, plates and shells or combinations thereof are widely used. Examples include trusses, frames, robot arms, solar panels, antennae, deformable mirrors, etc. We refer to [1] for more applications of flexible structure networks. For a network of complex elastic structures, not only its global dynamic motion has to be taken into account, but also the flexibility of individual elements as well as the interaction and transmission of elastic effects such as bending, torsion and axial deformations at the nodal

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points. The mathematical theory of distributed parameter system control models plays an important role in the study of such multi-link flexible structure networks.

This theory has been largely developed in the past three decades: The first study of control problems for flexible structure networks can be traced back to the 1970s. A result of Rolewicz [2] says that if the number of strings is greater than the number of knots, then the system is not exactly controllable. Later on, in the 1980s, Chen *et al.* [3] dealt with the stabilization of serially connected beams by means of the energy multiplier method; four possible configurations of joints for coupled beams with dissipativity were set up in [4]. From the beginning of the 1990s, more and more mathematicians and engineers have become involved in the study of flexible structure networks based on distributed parameter models. Among them, Lagnese *et al.* [5] made substantial contributions to the modelling and controllability of multi-link flexible beam and string structures: first, by virtue of Hamilton's principle, Schmidt derived a nonlinear system of partial differential equations for networks of vibrating strings [6] and obtained a controllability result for the linearized coupled wave equations. Using an analogous approach, Lagnese *et al.* [5] considered the same problem for networks of thin beams. In [7], they described a general linear model for planar/non-planar vibrating networks of elastic strings or Timoshenko beams by means of a system of coupled hyperbolic equations. Recently, Dekoninck and Nicaise [8] studied the exact controllability problem for networks of hyperbolic systems governed by Euler–Bernoulli beams with boundary dampings. They obtained some sufficient conditions using the multiplier method and Ingham's inequality. Furthermore, in [9], they proved some results on the spectrum; in particular, they showed that the spectrum depends only on the structure of the graph. References to some other recent works in this area may be found in [10] where the controllability problem for controlled tree-shaped networks of vibrating elastic strings was investigated and some relations between the traces of the solutions at the nodes of the network were found.

All these results form a solid basis to pursue higher-level research on the control and stabilization of flexible structure networks by hard analysis methods such as the Riesz basis approach developed in non-harmonic Fourier analysis, which has been used successfully in the study of dynamics and control of vibration of flexible systems (see, e.g. [11–20]).

In this paper, we study the stabilization of a symmetric tree-shaped multi-link flexible structures with four knots P_0, P_1, P_2 and P_3 sketched in Figure 1 with $\ell_1=\ell_2=\ell_3=1$. Assume that the controls $u_i(t)$ are proposed at the knot P_i for $1\leqslant i\leqslant 3$, respectively. The special feature arising here is the system of coupled partial differential equations and pointwise controls and observations. Since the linear operators describing such systems are non-self-adjoint, the Riesz basis generation becomes the most profound result for such a system. Three issues are addressed: (a) the expansion of solution in terms of the eigenfunctions and associated functions of the closed-loop system; (b) the asymptotic expansion of the eigenfrequencies; and (c) the spectrum-determined growth condition and the stability of the closed-loop system.

The corresponding dynamic system is given on the region $Q:=\{(x,t);\ 0<x<1,\ t>0\}$ by

$$\begin{cases} y_{tt}(x,t)+y_{xxxx}(x,t)=0 \\ y(1,t)=w(1,t) \\ y_x(1,t)=w_x(1,t) \\ y_{xx}(0,t)=0 \\ y_{xxx}(0,t)=u_1(t) \end{cases} \qquad \begin{cases} w_{tt}(x,t)+w_{xxxx}(x,t)=0 \\ w(1,t)=z(1,t) \\ w_x(1,t)=z_x(1,t) \\ w_{xx}(0,t)=0 \\ w_{xxx}(0,t)=u_2(t) \end{cases} \tag{1}$$

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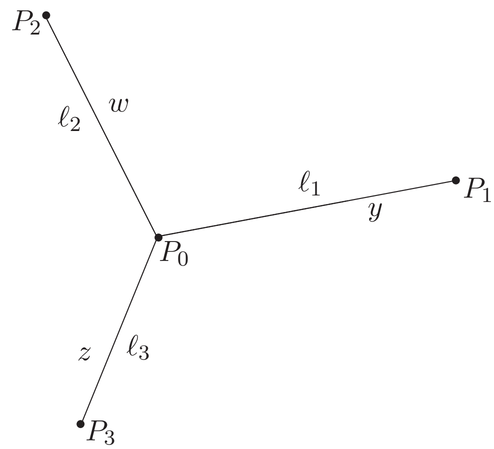

Figure 1. A tree-shaped network of Euler–Bernoulli beams.

and

$$\begin{cases} z_{tt}(x,t) + z_{xxxx}(x,t) = 0 \\ z_{xx}(0,t) = 0 \\ y_{xx}(1,t) + w_{xx}(1,t) + z_{xx}(1,t) = 0 \\ z_{xxx}(0,t) = u_3(t) \\ y_{xxx}(1,t) + w_{xxx}(1,t) + z_{xxx}(1,t) = 0 \end{cases} \tag{2}$$

The total energy of system (1)–(2) is given by

$$E(t) = \frac{1}{2}\int_0^1 [y_t^2(x,t) + y_{xx}^2(x,t) + w_t^2(x,t) + w_{xx}^2(x,t) + z_t^2(x,t) + z_{xx}^2(x,t)]\,\mathrm{d}x \tag{3}$$

Differentiate formally the energy function with respect to time t to give

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) = y_t(0,t)y_{xxx}(0,t) + w_t(0,t)w_{xxx}(0,t) + z_t(0,t)z_{xxx}(0,t) \tag{4}$$

So a simple control can be designed as

$$u_1(t) = -ky_t(0,t), \quad u_2(t) = -kw_t(0,t), \quad u_3(t) = -kz_t(0,t) \tag{5}$$

where k is a positive constant, which results in

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) = -ky_t^2(0,t) - kw_t^2(0,t) - kz_t^2(0,t) \leqslant 0 \tag{6}$$

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By the above design, the closed-loop form of system (1)–(5) becomes

$$\begin{cases} y_{tt}(x,t)+y_{xxxx}(x,t)=0 \\ y(1,t)=w(1,t) \\ y_x(1,t)=w_x(1,t) \\ y_{xx}(0,t)=0 \\ y_{xxx}(0,t)=-ky_t(0,t) \end{cases} \quad \begin{cases} w_{tt}(x,t)+w_{xxxx}(x,t)=0 \\ w(1,t)=z(1,t) \\ w_x(1,t)=z_x(1,t) \\ w_{xx}(0,t)=0 \\ w_{xxx}(0,t)=-kw_t(0,t) \end{cases} \tag{7}$$

and

$$\begin{cases} z_{tt}(x,t)+z_{xxxx}(x,t)=0 \\ z_{xx}(0,t)=0 \\ y_{xx}(1,t)+w_{xx}(1,t)+z_{xx}(1,t)=0 \\ z_{xxx}(0,t)=-kz_t(0,t) \\ y_{xxx}(1,t)+w_{xxx}(1,t)+z_{xxx}(1,t)=0 \end{cases} \tag{8}$$

The main objective of this paper is dedicated to give a comprehensive analysis for the dynamic behaviour of system (7)–(8) *via* spectral analysis.

2. BASIC PROPERTIES OF SYSTEM OPERATOR

We consider system (7)–(8) in the state Hilbert space $\mathscr{H}$

$$\mathscr{H}:=\{Y=[f_1,g_1,f_2,g_2,f_3,g_3]\in(H^2(0,1)\times L^2(0,1))^3;\ f_1^{(i)}(1)=f_2^{(i)}(1)=f_3^{(i)}(1),\ i=0,1\} \tag{9}$$

equipped with the norm induced by the inner product

$$\|Y\|=\int_0^1[|f_1''(x)|^2+|g_1(x)|^2+|f_2''(x)|^2+|g_2(x)|^2 + |f_3''(x)|^2+|g_3(x)|^2]\,\mathrm{d}x+K_1|f_1(0)|^2+K_2|f_2(0)|^2 \tag{10}$$

where K_1 and K_2 are two positive constants and the sign $'$ denotes the differentiation in spacial variable x. Define a linear operator $\mathscr{A}$ in $\mathscr{H}$ by

$$\mathscr{A}Y=[g_1,-f_1^{(4)},g_2,-f_2^{(4)},g_3,-f_3^{(4)}]\quad \forall Y=[f_1,g_1,f_2,g_2,f_3,g_3]\in D(\mathscr{A}) \tag{11}$$

with

$$D(\mathscr{A}):=\left\{\begin{array}{l} [f_1,g_1,f_2,g_2,f_3,g_3]\in(H^4(0,1)\times H^2(0,1))^3\cap\mathscr{H} \\ g_1^{(i)}(1)=g_2^{(i)}(1)=g_3^{(i)}(1),\ i=0,1,\ f_j''(0)=0,\ f_j'''(0)=-kg_j(0),\ j=1,2,3 \\ f_1''(1)+f_2''(1)+f_3''(1)=0,\ f_1'''(1)+f_2'''(1)+f_3'''(1)=0 \end{array}\right\} \tag{12}$$

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Let $Y(t) := [y(\cdot, t), y_t(\cdot, t), w(\cdot, t), w_t(\cdot, t), z(\cdot, t), z_t(\cdot, t)]$. Then system (7)–(8) can be formulated into the following abstract evolution equation in $\mathscr{H}$ with the initial data Y_0:

$$\frac{\mathrm{d}}{\mathrm{d}t} Y(t) = \mathscr{A} Y(t), \quad Y(0) = Y_0 \tag{13}$$

Lemma 2.1
Let $\mathscr{A}$ be defined by (11) and (12). Then 0 is an eigenvalue of $\mathscr{A}$ and its associated root subspace is

$$\mathfrak{E}(0, \mathscr{A}) := \mathrm{span}\{[1, 0, 1, 0, 1, 0],\ [x, 0, x, 0, x, 0],\ [0, x, 0, x, 0, x]\}$$

Proof
In order to prove that $0 \in \sigma(\mathscr{A})$, we only need to show that there is a

$$0 \neq F = [f_1, g_1, f_2, g_2, f_3, g_3] \in D(\mathscr{A})$$

so that $\mathscr{A} F = 0$, or equivalently, there is a non-zero solution to the following equation:

$$\begin{cases} g_1 = g_2 = g_3 = 0, \quad f_1^{(4)}(x) = f_2^{(4)}(x) = f_3^{(4)}(x) = 0 \\ f_1(1) = f_2(1) = f_3(1), \quad f_1'(1) = f_2'(1) = f_3'(1) \\ f_1''(0) = f_2''(0) = f_3''(0) = 0, \quad f_1'''(0) = f_2'''(0) = f_3'''(0) = 0 \\ f_1''(1) + f_2''(1) + f_3''(1) = 0, \quad f_1'''(1) + f_2'''(1) + f_3'''(1) = 0 \end{cases}$$

A direct computation shows that

$$f_i(x) = a_{0i} + a_{1i} x, \ \ i = 1, 2, 3, \quad a_{01} = a_{02} = a_{03}, \quad a_{11} = a_{12} = a_{13}$$

Thus 0 is an eigenvalue of $\mathscr{A}$ and

$$[1, 0, 1, 0, 1, 0] \quad \text{and} \quad [x, 0, x, 0, x, 0]$$

are two associated independent eigenfunctions. Furthermore, solve $\mathscr{A} Y = [x, 0, x, 0, x, 0]$ with $Y \in D(\mathscr{A})$ to produce a generalized eigenfunction $[0, x, 0, x, 0, x]$, which is the only one linearly independent generalized eigenfunction. The proof is complete. □

Lemma 2.2
$i \in \rho(\mathscr{A})$ and $\mathscr{A}$ is a densely defined operator in $\mathscr{H}$. Moreover, $(i - \mathscr{A})^{-1}$ is compact on $\mathscr{H}$. Therefore, the spectrum $\sigma(\mathscr{A})$ of $\mathscr{A}$ consists of isolated eigenvalues only.

Proof
We first show that $(i - \mathscr{A})^{-1}$ does exist. Let $F := [\varphi_1, \psi_1, \varphi_2, \psi_2, \varphi_3, \psi_3] \in \mathscr{H}$. Let us find $Y := [f_1, g_1, f_2, g_2, f_3, g_3] \in D(\mathscr{A})$ such that

$$(i - \mathscr{A}) Y = F$$

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This is equivalent to saying that

$$\begin{cases} if_1 - g_1 = \varphi_1, \quad ig_1 + f_1^{(4)} = \psi_1, \quad f_1'''(0) = -kg_1(0) \\ if_2 - g_2 = \varphi_2, \quad ig_2 + f_2^{(4)} = \psi_2, \quad f_2'''(0) = -kg_2(0) \\ if_3 - g_3 = \varphi_3, \quad ig_3 + f_3^{(4)} = \psi_3, \quad f_3'''(0) = -kg_3(0) \\ f_1(1) = f_2(1) = f_3(1), \quad f_1'(1) = f_2'(1) = f_3'(1) \\ f_1''(0) = f_2''(0) = f_3''(0) = 0 \\ f_1''(1) + f_2''(1) + f_3''(1) = 0, \quad f_1'''(1) + f_2'''(1) + f_3'''(1) = 0 \end{cases}$$

from which, we get

$$\begin{cases} g_1 = if_1 - \varphi_1, \quad g_2 = if_2 - \varphi_2, \quad g_3 = if_3 - \varphi_3 \\ f_1^{(4)} - f_1 = \psi_1 + i\varphi_1, \quad f_2^{(4)} - f_2 = \psi_2 + i\varphi_2, \quad f_3^{(4)} - f_3 = \psi_3 + i\varphi_3 \\ f_1(1) = f_2(1) = f_3(1), \quad f_1'(1) = f_2'(1) = f_3'(1) \\ f_1''(0) = f_2''(0) = f_3''(0) = 0, \quad f_1''(1) + f_2''(1) + f_3''(1) = 0 \\ f_1'''(0) = -ikf_1(0) + k\varphi_1(0), \quad f_2'''(0) = -ikf_2(0) + k\varphi_2(0) \\ f_3'''(0) = -ikf_3(0) + k\varphi_3(0), \quad f_1'''(1) + f_2'''(1) + f_3'''(1) = 0 \end{cases} \tag{14}$$

Suppose

$$f_i(x) = \widetilde{f}_i(x) + F_i(x), \quad i = 1, 2, 3$$

where $F_i(x)$ satisfy

$$\begin{cases} F_i^{(4)}(x) - F_i(x) = \psi_i(x) + i\varphi_i(x), \quad i = 1, 2, 3 \\ F_i(0) = F_i'(0) = F_i''(0) = F_i'''(0) = 0 \end{cases}$$

A particular solution of above equation is given by

$$F_i(x) = \frac{1}{2}\int_0^x [\sinh(x-\xi) - \sin(x-\xi)][\psi_i(\xi) + i\varphi_i(\xi)]\,\mathrm{d}\xi, \quad i = 1, 2, 3$$

It then follows from (14) that $\widetilde{f}_i(x), i = 1, 2, 3$, satisfy

$$\begin{cases} \widetilde{f}_1^{(4)} - \widetilde{f}_1 = 0, \quad \widetilde{f}_2^{(4)} - \widetilde{f}_2 = 0, \quad \widetilde{f}_3^{(4)} - \widetilde{f}_3 = 0 \\ \widetilde{f}_1(1) + F_1(1) = \widetilde{f}_2(1) + F_2(1) = \widetilde{f}_3(1) + F_3(1) \\ \widetilde{f}_1'(1) + F_1'(1) = \widetilde{f}_2'(1) + F_2'(1) = \widetilde{f}_3'(1) + F_3'(1) \\ \widetilde{f}_1''(1) + F_1''(1) + \widetilde{f}_2''(1) + F_2''(1) + \widetilde{f}_3''(1) + F_3''(1) = 0 \\ \widetilde{f}_1'''(0) = -ik\widetilde{f}_1(0) + k\varphi_1(0), \quad \widetilde{f}_2'''(0) = -ik\widetilde{f}_2(0) + k\varphi_2(0) \\ \widetilde{f}_3'''(0) = -ik\widetilde{f}_3(0) + k\varphi_3(0), \quad \widetilde{f}_1''(0) = \widetilde{f}_2''(0) = \widetilde{f}_3''(0) = 0 \\ \widetilde{f}_1'''(1) + F_1'''(1) + \widetilde{f}_2'''(1) + F_2'''(1) + \widetilde{f}_3'''(1) + F_3'''(1) = 0 \end{cases} \tag{15}$$

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For $\widetilde{f}_i$, $i=1,2,3$, the general solutions can be represented as

$$\widetilde{f}_i(x)=c_{1i}\sinh x+c_{2i}\sin x+c_{3i}(\cosh x+\cos x),\quad i=1,2,3$$

where c_{1i}, c_{2i} and c_{3i} are constants to be determined from boundary conditions of (15) that

$$\left\{\begin{array}{l}
(c_{11}-c_{12})\sinh 1+(c_{21}-c_{22})\sin 1+(c_{31}-c_{32})(\cosh 1+\cos 1)=F_2(1)-F_1(1)\\
(c_{11}-c_{13})\sinh 1+(c_{21}-c_{23})\sin 1+(c_{31}-c_{33})(\cosh 1+\cos 1)=F_3(1)-F_1(1)\\
(c_{11}-c_{12})\cosh 1+(c_{21}-c_{22})\cos 1+(c_{31}-c_{32})(\sinh 1-\sin 1)=F_2'(1)-F_1'(1)\\
(c_{11}-c_{13})\cosh 1+(c_{21}-c_{23})\cos 1+(c_{31}-c_{33})(\sinh 1-\sin 1)=F_3'(1)-F_1'(1)\\
(c_{11}+c_{12}+c_{13})\sinh 1-(c_{21}+c_{22}+c_{23})\sin 1+(c_{31}+c_{32}+c_{33})(\cosh 1-\cos 1)\\
\quad =-(F_1''(1)+F_2''(1)+F_3''(1))\\
c_{11}-c_{21}+2ikc_{31}=k\varphi_1(0),\quad c_{12}-c_{22}+2ikc_{32}=k\varphi_2(0)\\
c_{13}-c_{23}+2ikc_{33}=k\varphi_3(0)\\
(c_{11}+c_{12}+c_{13})\cosh 1-(c_{21}+c_{22}+c_{23})\cos 1+(c_{31}+c_{32}+c_{33})(\sinh 1+\sin 1)\\
\quad -ik_2(c_{11}\sinh 1+c_{21}\sin 1+c_{31}(\cosh 1+\cos 1))\\
\qquad =-(F_1'''(1)+F_2'''(1)+F_3'''(1))
\end{array}\right. \tag{16}$$

The uniqueness and existence of solution to (14) are equivalent to (16) for c_{1i}, c_{2i}, c_{3i}. In other words, the determinant Δ_c of the coefficient matrix of (16) is non-zero:

$$\Delta_c=\begin{vmatrix}
\sinh 1 & -\sinh 1 & 0 & \sin 1 & -\sin 1 & 0 & d_1 & -d_1 & 0\\
\sinh 1 & 0 & -\sinh 1 & \sin 1 & 0 & -\sin 1 & d_1 & 0 & -d_1\\
\cosh 1 & -\cosh 1 & 0 & \cos 1 & -\cos 1 & 0 & d_2 & -d_2 & 0\\
\cosh 1 & 0 & -\cosh 1 & \cos 1 & 0 & -\cos 1 & d_2 & 0 & -d_2\\
\sinh 1 & \sinh 1 & \sinh 1 & -\sin 1 & -\sin 1 & -\sin 1 & d_3 & d_3 & d_3\\
1 & 0 & 0 & -1 & 0 & 0 & 2ik & 0 & 0\\
0 & 1 & 0 & 0 & -1 & 0 & 0 & 2ik & 0\\
0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 2ik\\
\cosh 1 & \cosh 1 & \cosh 1 & -\cos 1 & -\cos 1 & -\cos 1 & d_4 & d_4 & d_4
\end{vmatrix} \tag{17}$$

where

$$d_1:=\cosh 1+\cos 1,\quad d_2:=\sinh 1-\sin 1,\quad d_3:=\cosh 1-\cos 1,\quad d_4:=\sinh 1+\sin 1$$

A direct computation shows that

$$\Delta_c=\Delta_{1c}^2(\Delta_{rc}+i\Delta_{ic})\neq 0$$

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where

$$\Delta_{1c} := \sinh^2 1 - \sin^2 1 - d_1^2 + 2ik(\sinh 1 \cos 1 - \cosh 1 \sin 1)$$

$$\Delta_{rc} := -9d_2d_4 + 3d_3^2, \quad \Delta_{ic} := 18kd_2 \cosh 1 - 6kd_3 \sinh 1$$

Thus (16) does possess a non-trivial solution and hence there exists a non-zero solution $\widetilde{f}_i(x)$, $i = 1, 2, 3$ to Equation (15). Therefore, there is a non-trivial element $Y \in D(\mathscr{A})$ such that $(i - \mathscr{A})Y = F$ and thus $(i - \mathscr{A})^{-1}$ exists.

Secondly, the Sobolev-embedding theorem ensures that $(i - \mathscr{A})^{-1}$ is compact on $\mathscr{H}$ and thus $\mathscr{A}$ is a discrete operator in $\mathscr{H}$, that is, there is a $\lambda \in \rho(\mathscr{A})$ such that $(\lambda - \mathscr{A})^{-1}$ is compact (see [21]). Therefore, the spectrum $\sigma(\mathscr{A})$ of $\mathscr{A}$ consists of only isolated eigenvalues with finite algebraic multiplicities. The proof is complete. □

Lemma 2.3

For each $\lambda \in \sigma(\mathscr{A})$ with $\lambda \neq 0$, we have $\text{Re}(\lambda) < 0$.

Proof

We first claim that $\text{Re}(\lambda) \leqslant 0$ for any $\lambda \in \sigma(\mathscr{A})$. Indeed, it is routine that $\lambda \in \sigma(\mathscr{A})$ if and only if λ satisfies the characteristic equations of the following:

$$\begin{cases} \lambda^2 y(x) + y^{(4)}(x) = 0 \\ y(1) = w(1) \\ y'(1) = w'(1) \\ y''(0) = 0 \\ y'''(0) = -k\lambda y(0) \end{cases} \quad \begin{cases} \lambda^2 w(x) + w^{(4)}(x) = 0 \\ w(1) = z(1) \\ w'(1) = z'(1) \\ w''(0) = 0 \\ w'''(0) = -k\lambda w(0) \end{cases} \tag{18}$$

and

$$\begin{cases} \lambda^2 z(x) + z^{(4)}(x) = 0 \\ z''(0) = 0 \\ y''(1) + w''(1) + z''(1) = 0 \\ z'''(0) = -k\lambda z(0) \\ y'''(1) + w'''(1) + z'''(1) = 0 \end{cases} \tag{19}$$

Take the inner products in $L^2(0, 1)$ with $y(x)$, $w(x)$ and $z(x)$, respectively, for both sides of above equations and take the boundary conditions into account, to obtain

$$\lambda^2 \|y\|_{L^2}^2 + \langle y^{(4)}, y \rangle_{L^2} = 0, \quad \lambda^2 \|w\|_{L^2}^2 + \langle w^{(4)}, w \rangle_{L^2} = 0, \quad \lambda^2 \|z\|_{L^2}^2 + \langle z^{(4)}, z \rangle_{L^2} = 0$$

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Invoke integration by parts to get

$$\begin{cases} \lambda^2\|y\|_{L^2}^2 + y'''(1)\overline{y(1)} + k\lambda|y(0)|^2 - y''(1)\overline{y'(1)} + \|y''\|_{L^2}^2 = 0 \\ \lambda^2\|w\|_{L^2}^2 + w'''(1)\overline{w(1)} + k\lambda|w(0)|^2 - w''(1)\overline{w'(1)} + \|w''\|_{L^2}^2 = 0 \\ \lambda^2\|z\|_{L^2}^2 + z'''(1)\overline{z(1)} + k\lambda|z(0)|^2 - z''(1)\overline{z'(1)} + \|z''\|_{L^2}^2 = 0 \end{cases}$$

Add three identities together obtain

$$\begin{aligned} &\lambda^2(\|y\|_{L^2}^2 + \|w\|_{L^2}^2 + \|z\|_{L^2}^2) \\ &\quad + k\lambda(|y(0)|^2 + |w(0)|^2 + |z(0)|^2) + \|y''\|_{L^2}^2 + \|w''\|_{L^2}^2 + \|z''\|_{L^2}^2 = 0 \end{aligned}$$

Let $\lambda = \text{Re}(\lambda) + i\,\text{Im}(\lambda)$. Then, it follows that

$$\begin{aligned} &((\text{Re}(\lambda))^2 - (\text{Im}(\lambda))^2)(\|y\|_{L^2}^2 + \|w\|_{L^2}^2 + \|z\|_{L^2}^2) \\ &\quad + k\,\text{Re}(\lambda)(|y(0)|^2 + |w(0)|^2 + |z(0)|^2) + \|y''\|_{L^2}^2 + \|w''\|_{L^2}^2 + \|z''\|_{L^2}^2 = 0 \end{aligned} \tag{20}$$

and

$$2\,\text{Re}(\lambda)\text{Im}(\lambda)(\|y\|_{L^2}^2 + \|w\|_{L^2}^2 + \|z\|_{L^2}^2) + k\,\text{Im}(\lambda)(|y(0)|^2 + |w(0)|^2 + |z(0)|^2) = 0 \tag{21}$$

By (21), if $\text{Im}(\lambda) \neq 0$, then it has

$$\text{Re}(\lambda) = -\frac{k(|y(0)|^2 + |w(0)|^2 + |z(0)|^2)}{2(\|y\|_{L^2}^2 + \|w\|_{L^2}^2 + \|z\|_{L^2}^2)} \leqslant 0$$

Otherwise, if $\text{Im}(\lambda) = 0$, then, by (20), it has

$$\text{Re}(\lambda) \leqslant 0$$

Hence in any case, it always has $\text{Re}(\lambda) \leqslant 0$.

Next, we show that $\text{Re}(\lambda) < 0$ for any $\lambda \in \sigma(\mathscr{A})$ whenever $\lambda \neq 0$. Let $\lambda = i\tau^2$, $0 \neq \tau \in \mathbb{R}$. Then, (18)–(19) becomes

$$\begin{cases} y^{(4)}(x) - \tau^4 y(x) = 0 \\ y(1) = w(1) \\ y'(1) = w'(1) \\ y''(0) = 0 \\ y'''(0) = -ik\tau^2 y(0) \end{cases} \qquad \begin{cases} w^{(4)}(x) - \tau^4 w(x) = 0 \\ w(1) = z(1) \\ w'(1) = z'(1) \\ w''(0) = 0 \\ w'''(0) = -ik\tau^2 w(0) \end{cases} \tag{22}$$

and

$$\begin{cases} z^{(4)}(x) - \tau^4 z(x) = 0 \\ z''(0) = 0 \\ y''(1) + w''(1) + z''(1) = 0 \\ z'''(0) = -ik\tau^2 z(0) \\ y'''(1) + w'''(1) + z'''(1) = 0 \end{cases} \tag{23}$$

By (21), we also have

$$y(1) = y(0) = w(0) = z(0) = 0 \tag{24}$$

that further makes (22) and (23) into the following forms:

$$\begin{cases} y^{(4)}(x) - \tau^4 y(x) = 0 \\ y'''(0) = y''(0) = y(0) = 0 \\ y(1) = 0, \quad y'(1) = w'(1) \end{cases} \qquad \begin{cases} w^{(4)}(x) - \tau^4 w(x) = 0 \\ w'''(0) = w''(0) = w(0) = 0 \\ w(1) = 0, \quad w'(1) = z'(1) \end{cases} \tag{25}$$

and

$$\begin{cases} z^{(4)}(x) - \tau^4 z(x) = 0 \\ z'''(0) = z''(0) = z(0) = z(1) = 0 \\ y''(1) + w''(1) + z''(1) = 0 \\ y'''(1) + w'''(1) + z'''(1) = 0 \end{cases} \tag{26}$$

By the boundary conditions $y'''(0) = y''(0) = y(0) = 0$, $w'''(0) = w''(0) = w(0) = 0$ and $z'''(0) = z''(0) = z(0) = 0$, the solutions of (25)–(26) can be represented as

$$y(x) = c_1(\sinh \tau x + \sin \tau x), \quad w(x) = c_2(\sinh \tau x + \sin \tau x), \quad z(x) = c_3(\sinh \tau x + \sin \tau x)$$

By conditions $y(1) = w(1) = z(1) = 0$ and $y''(1) + w''(1) + z''(1) = 0$, it is seen that $c_1 = c_2 = c_3 = 0$. So, there are only zero solutions to Equation (22)–(23) and hence $\lambda = i\tau^2, 0 \neq \tau \in \mathbb{R}$ is not eigenvalue of $\mathscr{A}$. Therefore, $\mathrm{Re}(\lambda) < 0$ for each $\lambda \in \sigma(\mathscr{A})$ whenever $\lambda \neq 0$. The proof is complete. □

Lemma 2.4
$\mathscr{A}$ generates a C_0-semigroup $e^{\mathscr{A}t}$ on $\mathscr{H}$.

Proof
For given $Y = [f_1, g_1, f_2, g_2, f_3, g_3] \in D(\mathscr{A})$, we have

$$\begin{aligned} \langle \mathscr{A}Y, Y \rangle &= \langle [g_1, -f_1^{(4)}, g_2, -f_2^{(4)}, g_3, -f_3^{(4)}], Y \rangle \\ &= \int_0^1 [g_1'' \overline{f_1''} - f_1^{(4)} \overline{g_1} + g_2'' \overline{f_2''} - f_2^{(4)} \overline{g_2} + g_3'' \overline{f_3''} - f_3^{(4)} \overline{g_3}] \, \mathrm{d}x \end{aligned}$$

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$$
\begin{aligned}
&+K_1 g_1(0)\overline{f_1(0)} + K_2 g_2(0)\overline{f_2(0)} \\
&= \int_0^1 [g_1''\overline{f_1''} - f_1''\overline{g_1''} + g_2''\overline{f_2''} - f_2''\overline{g_2''} + g_3''\overline{f_3''} - f_3''\overline{g_3''}]\,\mathrm{d}x \\
&-k|g_1(0)|^2 - k|g_2(0)|^2 - k|g_3(0)|^2 + K_1 g_1(0)\overline{f_1(0)} + K_2 g_2(0)\overline{f_2(0)}
\end{aligned}
$$

hence

$$
\begin{aligned}
\mathrm{Re}\langle \mathscr{A}Y, Y\rangle &= -k|g_1(0)|^2 - k|g_2(0)|^2 - k|g_3(0)|^2 + K_1\,\mathrm{Re}(g_1(0)\overline{f_1(0)}) + K_2\,\mathrm{Re}(g_2(0)\overline{f_2(0)}) \\
&\leqslant -k|g_1(0)|^2 - k|g_2(0)|^2 - k|g_3(0)|^2 + K_1\frac{\varepsilon}{2}|g_1(0)|^2 + K_1\frac{1}{2\varepsilon}|f_1(0)|^2 \\
&\quad + K_2\frac{\varepsilon}{2}|g_2(0)|^2 + K_1\frac{1}{2\varepsilon}|f_2(0)|^2 \\
&\leqslant \left(K_1\frac{\varepsilon}{2} - k\right)|g_1(0)|^2 + \left(K_2\frac{\varepsilon}{2} - k\right)|g_2(0)|^2 - k|g_3(0)|^2 + K_1\frac{1}{2\varepsilon}\|Y\|^2
\end{aligned}
$$

Choosing ε sufficiently small so that

$$
K_1\frac{\varepsilon}{2} - k<0, \qquad K_2\frac{\varepsilon}{2} - k<0
$$

we get

$$
\mathrm{Re}\langle \mathscr{A}Y, Y\rangle \leqslant K_1\frac{1}{2\varepsilon}\|Y\|^2
$$

Therefore, $\mathscr{A} - K_1(2\varepsilon)^{-1}$ is a dissipative operator in $\mathscr{H}$. By Lemma 2.2, it follows that $(i - \mathscr{A})^{-1}$ exists and is compact on $\mathscr{H}$, so is for $(i + K_1(2\varepsilon)^{-1} - \mathscr{A})^{-1}$. Hence, by the Lumer–Phillips theorem [22], $\mathscr{A} - K_1(2\varepsilon)^{-1}$ generates a C_0-semigroup of contractions on $\mathscr{H}$. So does $\mathscr{A}$. The proof is complete. □

3. ASYMPTOTIC BEHAVIOUR OF EIGENFREQUENCIES

In this section, we analyse the spectral behaviour of system (7)–(8). The standard technique due to Birkhoff and Langer (see, [23]) and Tretter (see [24, 25]) is under consideration. From the characteristic equation (18)–(19), we know that the spectrum of $\mathscr{A}$ is distributed symmetrically about the real axis. Due to this property, it is easy to find that for a non-zero solution $(y(x), w(x), z(x))$ to (18)–(19) with respect to an eigenvalue λ

$$
(w(x), z(x), y(x)), \qquad (z(x), y(x), w(x))
$$

are also non-trivial solutions to (18)–(19). The following Lemma 3.1 is straightforward.

Lemma 3.1
Let $(y(x), w(x), z(x))$ be a solution to (18)–(19). Then $(w(x), z(x), y(x))$ and $(z(x), y(x), w(x))$ are also solutions of (18)–(19) and there are at most three independent solutions to (18)–(19).

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Now we investigate the eigenvalue problem of system (7)–(8). To this purpose, we first transform the characteristic equation (18)–(19) into a system of first-order differential equation parameterized by eigenvalue λ. Actually, let

$$\begin{cases} \Phi(\cdot) := [\varphi_1^y, \varphi_2^y, \varphi_3^y, \varphi_4^y, \varphi_1^w, \varphi_2^w, \varphi_3^w, \varphi_4^w, \varphi_1^z, \varphi_2^z, \varphi_3^z, \varphi_4^z]^\top \\ \varphi_1^y = y, \quad \varphi_2^y = y', \quad \varphi_3^y = y'', \quad \varphi_4^y = y''' \\ \varphi_1^w = w, \quad \varphi_2^w = w', \quad \varphi_3^w = w'', \quad \varphi_4^w = w''' \\ \varphi_1^z = z, \quad \varphi_2^z = z', \quad \varphi_3^z = z'', \quad \varphi_4^z = z''' \end{cases} \tag{27}$$

Then (18)–(19) is reformulated as

$$\begin{cases} T^D(x, \lambda)\Phi(x) := \Phi'(x) + A(\lambda)\Phi(x) = 0 \\ T^R\Phi(x) := W^0(\lambda)\Phi(0) + W^1(\lambda)\Phi(1) = 0 \end{cases} \tag{28}$$

where

$$A(\lambda) := \begin{bmatrix} M(\lambda) & & \\ & M(\lambda) & \\ & & M(\lambda) \end{bmatrix}, \quad M(\lambda) := \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ \lambda^2 & 0 & 0 & 0 \end{bmatrix} \tag{29}$$

$$W^0(\lambda) := \begin{bmatrix} W_1^0(\lambda) & & \\ & W_1^0(\lambda) & \\ & & W_1^0(\lambda) \\ O_{6\times 4} & O_{6\times 4} & O_{6\times 4} \end{bmatrix}, \quad W^1(\lambda) := \begin{bmatrix} O_{6\times 4} & O_{6\times 4} & O_{6\times 4} \\ W_1^1 & -W_1^1 & \\ & W_1^1 & -W_1^1 \\ W_2^1 & W_2^1 & W_2^1 \end{bmatrix} \tag{30}$$

with

$$\begin{cases} W_1^0(\lambda) := \begin{bmatrix} 0 & 0 & 1 & 0 \\ k\lambda & 0 & 0 & 1 \end{bmatrix}, \quad W_1^1 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \\ W_2^1 := \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{cases} \tag{31}$$

Lemma 3.2
Equation (18)–(19) is equivalent to the system of the parameterized first-order differential equation (28). Moreover, $0 \neq \lambda \in \sigma(\mathscr{A})$ if and only if there is a non-trivial solution Φ to (28).

We are now focusing on problem (28). Due to Lemma 2.3 and the fact that the eigenvalues are symmetric about the real axis, we consider only those λ that are located in the second quadrant of the complex plane

$$\lambda := i\rho^2, \quad \rho \in \mathscr{S} := \left\{ \rho \in \mathbb{C} \;\middle|\; 0 \leqslant \arg \rho \leqslant \frac{\pi}{4} \right\}$$

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Note that for any $\rho\in\mathscr{S}$, it has

$$\mathrm{Re}(-\rho)\leqslant\mathrm{Re}(i\rho)\leqslant\mathrm{Re}(-i\rho)\leqslant\mathrm{Re}(\rho)$$

and

$$\begin{cases}\mathrm{Re}(-\rho)=-|\rho|\cos(\arg\rho)\leqslant-\dfrac{\sqrt{2}}{2}|\rho|<0\\ \mathrm{Re}(i\rho)=-|\rho|\sin(\arg\rho)\leqslant 0\end{cases}$$

Define an invertible matrix function $P(\rho)$ for $\rho\in\mathbb{C}$, $\rho\neq 0$

$$P(\rho):=\begin{bmatrix}P_1(\rho)&&\\&P_1(\rho)&\\&&P_1(\rho)\end{bmatrix},\quad P_1(\rho):=\begin{bmatrix}\rho&\rho&\rho&\rho\\ \rho^2&-\rho^2&i\rho^2&-i\rho^2\\ \rho^3&\rho^3&-\rho^3&-\rho^3\\ \rho^4&-\rho^4&-i\rho^4&i\rho^4\end{bmatrix}\tag{32}$$

Thus, we have

$$P^{-1}(\rho)=\begin{bmatrix}P_1^{-1}(\rho)&&\\&P_1^{-1}(\rho)&\\&&P_1^{-1}(\rho)\end{bmatrix},\quad P_1^{-1}(\rho)=\begin{bmatrix}\dfrac{1}{4\rho}&\dfrac{1}{4\rho^2}&\dfrac{1}{4\rho^3}&\dfrac{1}{4\rho^4}\\ \dfrac{1}{4\rho}&-\dfrac{1}{4\rho^2}&\dfrac{1}{4\rho^3}&-\dfrac{1}{4\rho^4}\\ \dfrac{1}{4\rho}&-i\dfrac{1}{4\rho^2}&-\dfrac{1}{4\rho^3}&i\dfrac{1}{4\rho^4}\\ \dfrac{1}{4\rho}&i\dfrac{1}{4\rho^2}&-\dfrac{1}{4\rho^3}&-i\dfrac{1}{4\rho^4}\end{bmatrix}$$

Define a linear transformation to (28) with $\lambda=\rho^2$

$$\Psi(x):=P^{-1}(\rho)\Phi(x),\quad \widehat{T}^D(x,\rho):=P(\rho)^{-1}T^D(x,i\rho^2)P(\rho)\tag{33}$$

Then, we have

$$\widehat{T}^D(x,\rho)\Psi(x)=\Psi'(x)+\widehat{A}(\rho)\Psi(x)=0\tag{34}$$

where

$$\widehat{A}(\rho):=P(\rho)^{-1}A(i\rho^2)P(\rho)$$
$$=\begin{bmatrix}P_1^{-1}(\rho)&&\\&P_1^{-1}(\rho)&\\&&P_1^{-1}(\rho)\end{bmatrix}\begin{bmatrix}M(i\rho^2)&&\\&M(i\rho^2)&\\&&M(i\rho^2)\end{bmatrix}\begin{bmatrix}P_1(\rho)&&\\&P_1(\rho)&\\&&P_1(\rho)\end{bmatrix}$$

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$$= \begin{bmatrix} P_1^{-1}(\rho)M(i\rho^2)P_1(\rho) & & \\ & P_1^{-1}(\rho)M(i\rho^2)P_1(\rho) & \\ & & P_1^{-1}(\rho)M(i\rho^2)P_1(\rho) \end{bmatrix}$$

$$= \begin{bmatrix} \widehat{A}_1(\rho) & & \\ & \widehat{A}_1(\rho) & \\ & & \widehat{A}_1(\rho) \end{bmatrix}$$

with

$$\widehat{A}_1(\rho) := P_1^{-1}(\rho)M(i\rho^2)P_1(\rho) = \begin{bmatrix} -\rho & & & \\ & \rho & & \\ & & -i\rho & \\ & & & i\rho \end{bmatrix} \tag{35}$$

By these computations, it is seen that $\widehat{A}(\rho)$ is diagonal in ρ. Hence, it is easy to get the fundamental matrix solution to (34).

Lemma 3.3
Let $0 \neq \rho \in \mathscr{S}$. For $x \in [0, 1]$, there exists a fundamental matrix solution to the problem (34), which is given by

$$E(x,\rho) := \begin{bmatrix} E_1(x,\rho) & & \\ & E_1(x,\rho) & \\ & & E_1(x,\rho) \end{bmatrix}, \quad E_1(x,\rho) := \begin{bmatrix} e^{\rho x} & & & \\ & e^{-\rho x} & & \\ & & e^{i\rho x} & \\ & & & e^{-i\rho x} \end{bmatrix} \tag{36}$$

By (33), we know that

$$\widehat{\Phi}(x,\rho) := P(\rho)E(x,\rho) \tag{37}$$

is also a fundamental matrix solution to the first-order system (28) with $\lambda = i\rho^2$, $\rho \in \mathscr{S}$.

We are now ready to estimate asymptotically the distribution of eigenvalues of $\mathscr{A}$. From (28), $\lambda = i\rho^2 \in \sigma(\mathscr{A})$ for $\rho \in \mathscr{S}$ if and only if ρ is a zero of the characteristic determinant $\Delta(\rho)$

$$\Delta(\rho) := \det(T^R(i\rho^2)\widehat{\Phi}), \quad \rho \in \mathscr{S} \tag{38}$$

where the operator T^R is defined in (28) and $\widehat{\Phi}$ is the fundamental matrix solution given by (37). Since

$$T^R(i\rho^2)\widehat{\Phi} = W^0(i\rho^2)P(\rho)E(0,\rho) + W^1(i\rho^2)P(\rho)E(1,\rho) \tag{39}$$

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it follows from (30) and (31) that

$$W^0(i\rho^2)P(\rho)E(0,\rho) = \begin{bmatrix} \widehat{W}_1^0(\rho) & & \\ & \widehat{W}_1^0(\rho) & \\ & & \widehat{W}_1^0(\rho) \\ O_{6\times 4} & O_{6\times 4} & O_{6\times 4} \end{bmatrix}$$

where

$$\widehat{W}_1^0(\rho) := W_1^0(i\rho^2)P_1(\rho) = \begin{bmatrix} \rho^3 & \rho^3 & -\rho^3 & -\rho^3 \\ \rho^4\left(1+\dfrac{ik}{\rho}\right) & \rho^4\left(-1+\dfrac{ik}{\rho}\right) & i\rho^4\left(-1+\dfrac{k}{\rho}\right) & i\rho^4\left(1+\dfrac{k}{\rho}\right) \end{bmatrix}$$

$$W^1(i\rho^2)P(\rho)E(1,\rho) = \begin{bmatrix} O_{6\times 4} & O_{6\times 4} & O_{6\times 4} \\ \widehat{W}_1^1(\rho) & -\widehat{W}_1^1(\rho) & \\ & \widehat{W}_1^1(\rho) & -\widehat{W}_1^1(\rho) \\ \widehat{W}_2^1(\rho) & \widehat{W}_2^1(\rho) & \widehat{W}_2^1(\rho) \end{bmatrix}$$

with

$$\begin{cases} \widehat{W}_1^1(\rho) := W_1^1 P_1(\rho)E(1,\rho) = \begin{bmatrix} \rho e^{\rho} & \rho e^{-\rho} & \rho e^{i\rho} & \rho e^{-i\rho} \\ \rho^2 e^{\rho} & -\rho^2 e^{-\rho} & i\rho^2 e^{i\rho} & -i\rho^2 e^{-i\rho} \end{bmatrix} \\ \widehat{W}_2^1(\rho) := W_2^1 P_1(\rho)E(1,\rho) = \begin{bmatrix} \rho^3 e^{\rho} & \rho^3 e^{-\rho} & -\rho^3 e^{i\rho} & -\rho^3 e^{-i\rho} \\ \rho^4 e^{\rho} & -\rho^4 e^{-\rho} & -i\rho^4 e^{i\rho} & i\rho^4 e^{-i\rho} \end{bmatrix} \end{cases}$$

Hence

$$T^R(i\rho^2)\widehat{\Phi} = \begin{bmatrix} \widehat{W}_1^0(\rho) & & \\ & \widehat{W}_1^0(\rho) & \\ & & \widehat{W}_1^0(\rho) \\ \widehat{W}_1^1(\rho) & -\widehat{W}_1^1(\rho) & \\ & \widehat{W}_1^1(\rho) & -\widehat{W}_1^1(\rho) \\ \widehat{W}_2^1(\rho) & \widehat{W}_2^1(\rho) & \widehat{W}_2^1(\rho) \end{bmatrix} \tag{40}$$

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Therefore,

$$\Delta(\rho) = \det(T^R(i\rho^2)\widehat{\Phi}) = \det\begin{bmatrix} \widehat{W}_1^0(\rho) & & \\ & \widehat{W}_1^0(\rho) & \\ & & \widehat{W}_1^0(\rho) \\ \widehat{W}_1^1(\rho) & -\widehat{W}_1^1(\rho) & \\ & & -\widehat{W}_1^1(\rho) \\ \widehat{W}_2^1(\rho) & 2\widehat{W}_2^1(\rho) & \widehat{W}_2^1(\rho) \end{bmatrix}$$

$$= \det\begin{bmatrix} \widehat{W}_1^0(\rho) \\ -\widehat{W}_1^1(\rho) \end{bmatrix} \det\begin{bmatrix} \widehat{W}_1^0(\rho) & \\ & \widehat{W}_1^0(\rho) \\ \widehat{W}_1^1(\rho) & -\widehat{W}_1^1(\rho) \\ \widehat{W}_2^1(\rho) & 2\widehat{W}_2^1(\rho) \end{bmatrix}$$

$$= \det\begin{bmatrix} \widehat{W}_1^0(\rho) \\ -\widehat{W}_1^1(\rho) \end{bmatrix} \det\begin{bmatrix} \widehat{W}_1^0(\rho) & \\ & \widehat{W}_1^0(\rho) \\ & -\widehat{W}_1^1(\rho) \\ 3\widehat{W}_2^1(\rho) & 2\widehat{W}_2^1(\rho) \end{bmatrix}$$

$$= \left(\det\begin{bmatrix} \widehat{W}_1^0(\rho) \\ -\widehat{W}_1^1(\rho) \end{bmatrix}\right)^2 \det\begin{bmatrix} \widehat{W}_1^0(\rho) \\ 3\widehat{W}_2^1(\rho) \end{bmatrix}$$

$$= \left(\det\begin{bmatrix} \rho^3 & \rho^3 & -\rho^3 & -\rho^3 \\ \rho^4\left(1+\dfrac{ik}{\rho}\right) & \rho^4\left(-1+\dfrac{ik}{\rho}\right) & i\rho^4\left(-1+\dfrac{k}{\rho}\right) & i\rho^4\left(1+\dfrac{k}{\rho}\right) \\ \rho e^{\rho} & \rho e^{-\rho} & \rho e^{i\rho} & \rho e^{-i\rho} \\ \rho^2 e^{\rho} & -\rho^2 e^{-\rho} & i\rho^2 e^{i\rho} & -i\rho^2 e^{-i\rho} \end{bmatrix}\right)^2$$

$$\times \det\begin{bmatrix} \rho^3 & \rho^3 & -\rho^3 & -\rho^3 \\ \rho^4\left(1+\dfrac{ik}{\rho}\right) & \rho^4\left(-1+\dfrac{ik}{\rho}\right) & i\rho^4\left(-1+\dfrac{k}{\rho}\right) & i\rho^4\left(1+\dfrac{k}{\rho}\right) \\ 3\rho^3 e^{\rho} & 3\rho^3 e^{-\rho} & -3\rho^3 e^{i\rho} & -3\rho^3 e^{-i\rho} \\ 3\rho^4 e^{\rho} & -3\rho^4 e^{-\rho} & -3i\rho^4 e^{i\rho} & 3i\rho^4 e^{-i\rho} \end{bmatrix}$$

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$$= 9\rho^{34}\left(\det\begin{bmatrix} 1 & 1 & -1 & -1 \\ 1+\dfrac{ik}{\rho} & -1+\dfrac{ik}{\rho} & i\left(-1+\dfrac{k}{\rho}\right) & i\left(1+\dfrac{k}{\rho}\right) \\ e^{\rho} & e^{-\rho} & e^{i\rho} & e^{-i\rho} \\ e^{\rho} & -e^{-\rho} & ie^{i\rho} & -ie^{-i\rho} \end{bmatrix}\right)^2$$

$$\times\det\begin{bmatrix} 1 & 1 & -1 & -1 \\ 1+\dfrac{ik}{\rho} & -1+\dfrac{ik}{\rho} & i\left(-1+\dfrac{k}{\rho}\right) & i\left(1+\dfrac{k}{\rho}\right) \\ e^{\rho} & e^{-\rho} & -e^{i\rho} & -e^{-i\rho} \\ e^{\rho} & -e^{-\rho} & -ie^{i\rho} & ie^{-i\rho} \end{bmatrix}$$

$$= 9\rho^{34}e^{3\rho}[\mathcal{O}(e^{-c|\rho|}) + \Delta_1^2(\rho)\Delta_2(\rho)]$$

where $c>0$ is some positive constant and

$$\Delta_1(\rho) := \det\begin{bmatrix} 0 & 1 & -1 & -1 \\ 0 & -1+\dfrac{ik}{\rho} & i\left(-1+\dfrac{k}{\rho}\right) & i\left(1+\dfrac{k}{\rho}\right) \\ 1 & 0 & e^{i\rho} & e^{-i\rho} \\ 1 & 0 & ie^{i\rho} & -ie^{-i\rho} \end{bmatrix}$$

$$= \det\begin{bmatrix} 1 & -1 \\ -1+\dfrac{ik}{\rho} & i\left(-1+\dfrac{k}{\rho}\right) \end{bmatrix}\det\begin{bmatrix} 1 & e^{-i\rho} \\ 1 & -ie^{-i\rho} \end{bmatrix}$$

$$-\det\begin{bmatrix} 1 & -1 \\ -1+\dfrac{ik}{\rho} & i\left(1+\dfrac{k}{\rho}\right) \end{bmatrix}\det\begin{bmatrix} 1 & e^{i\rho} \\ 1 & ie^{i\rho} \end{bmatrix}$$

$$= -\left(-i+\frac{ik}{\rho}-1+\frac{ik}{\rho}\right)(e^{-i\rho}+ie^{-i\rho}) - \left(i+\frac{ik}{\rho}-1+\frac{ik}{\rho}\right)(ie^{i\rho}-e^{i\rho})$$

$$= \left(2i-2(i-1)\frac{k}{\rho}\right)e^{-i\rho} + \left(2i+2(i+1)\frac{k}{\rho}\right)e^{i\rho}$$

$$= 2i\left[\left(1-(i+1)\frac{k}{\rho}\right)e^{-i\rho} + \left(1+(1-i)\frac{k}{\rho}\right)e^{i\rho}\right]$$

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$$\Delta_2(\rho) := \det\begin{bmatrix} 0 & 1 & -1 & -1 \\ 0 & -1+\dfrac{ik}{\rho} & i\left(-1+\dfrac{k}{\rho}\right) & i\left(1+\dfrac{k}{\rho}\right) \\ 1 & 0 & -\mathrm{e}^{\mathrm{i}\rho} & -\mathrm{e}^{-\mathrm{i}\rho} \\ 1 & 0 & -i\mathrm{e}^{\mathrm{i}\rho} & i\mathrm{e}^{-\mathrm{i}\rho} \end{bmatrix}$$

$$= \det\begin{bmatrix} 1 & -1 \\ -1+\dfrac{ik}{\rho} & i\left(-1+\dfrac{k}{\rho}\right) \end{bmatrix}\det\begin{bmatrix} 1 & -\mathrm{e}^{-\mathrm{i}\rho} \\ 1 & i\mathrm{e}^{-\mathrm{i}\rho} \end{bmatrix}$$

$$- \det\begin{bmatrix} 1 & -1 \\ -1+\dfrac{ik}{\rho} & i\left(1+\dfrac{k}{\rho}\right) \end{bmatrix}\det\begin{bmatrix} 1 & -\mathrm{e}^{\mathrm{i}\rho} \\ 1 & -i\mathrm{e}^{\mathrm{i}\rho} \end{bmatrix}$$

$$= \left(-i+\frac{ik}{\rho}-1+\frac{ik}{\rho}\right)(\mathrm{e}^{-\mathrm{i}\rho}+i\mathrm{e}^{-\mathrm{i}\rho}) + \left(i+\frac{ik}{\rho}-1+\frac{ik}{\rho}\right)(i\mathrm{e}^{\mathrm{i}\rho}-\mathrm{e}^{\mathrm{i}\rho})$$

$$= -\Delta_1(\rho)$$

Theorem 3.1
Let $\Delta(\rho)$ be the characteristic determinant of system (28) in the sector $\mathscr{S}$ with $\lambda = i\rho^2$. Then the following asymptotic expansion holds

$$\Delta(\rho) = 72i\rho^{34}\mathrm{e}^{3\rho}\left[\mathcal{O}(\mathrm{e}^{-c|\rho|}) + \left(\left(1-(1+i)\frac{k}{\rho}\right)\mathrm{e}^{-\mathrm{i}\rho} + \left(1+(1-i)\frac{k}{\rho}\right)\mathrm{e}^{\mathrm{i}\rho}\right)^3\right] \tag{41}$$

where $c>0$ is some positive constant. Moreover, the algebraic multiplicity of all eigenvalues $\{0, \lambda_n, \overline{\lambda}_n\}$ of system (28) with sufficiently large modules is less than or equal to 9 and have the following asymptotic expansion:

$$\lambda_n = -2k + i\left(\frac{1}{2}+n\right)^2\pi^2 + \mathcal{O}(n^{-1}) \quad \text{as } n\to\infty \tag{42}$$

where n are positive integers. Therefore,

$$\mathrm{Re}\{\lambda_n, \overline{\lambda}_n\} \to -2k \quad \text{as } n\to\infty \tag{43}$$

Proof
Let $\Delta(\rho) = 0$, $\rho\in\mathscr{S}$. Equation (41) has already been proved. By (41), it follows that

$$\left(1-(i+1)\frac{k}{\rho}\right)\mathrm{e}^{-\mathrm{i}\rho} + \left(1+(1-i)\frac{k}{\rho}\right)\mathrm{e}^{\mathrm{i}\rho} + \mathcal{O}(\mathrm{e}^{-c|\rho|}) = 0 \tag{44}$$

This leads to

$$\mathrm{e}^{-\mathrm{i}\rho} + \mathrm{e}^{\mathrm{i}\rho} + \mathcal{O}(\rho^{-1}) = 0 \tag{45}$$

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Since in the first quadrant of the complex plane, the solutions of the equation

$$e^{i\rho} + e^{-i\rho} = 0$$

are given by

$$\tilde{\rho}_n = (\tfrac{1}{2} + n)\pi, \quad n = 0, 1, 2, \ldots$$

it follows from the Rouché's theorem that the solutions to Equation (45) are of the following form:

$$\rho_n = \tilde{\rho}_n + \alpha_n = (\tfrac{1}{2} + n)\pi + \alpha_n, \quad \alpha_n = \mathcal{O}(n^{-1}), \quad n = N, N+1, \ldots \tag{46}$$

where N is a sufficiently large positive integer. Substitute ρ_n into (44) and use the fact that $e^{i\tilde{\rho}_n} = -e^{-i\tilde{\rho}_n}$ to obtain

$$\left(1 + (1-i)\frac{k}{\rho_n}\right) e^{i\alpha_n} - \left(1 - (i+1)\frac{k}{\rho_n}\right) e^{-i\alpha_n} + \mathcal{O}(\rho^{-2}) = 0$$

Expand the exponential function into Taylor series to give

$$\alpha_n = -\frac{k}{i(\frac{1}{2} + n)\pi} + \mathcal{O}(n^{-2})$$

Substitute above into (46) to produce

$$\rho_n = \left(\frac{1}{2} + n\right)\pi - \frac{k}{i(\frac{1}{2} + n)\pi} + \mathcal{O}(n^{-2}) \quad \text{as } n \to \infty \tag{47}$$

Since $\lambda_n = i\rho_n^2$, we get eventually that

$$\lambda_n = -2k + i(\tfrac{1}{2} + n)^2\pi^2 + \mathcal{O}(n^{-1}) \quad \text{as } n \to \infty$$

Finally, we discuss the algebraic multiplicity of eigenvalues with large modulus. First, it follows from (41) and Rouché's theorem, the order of each zero of $\Delta(\rho)$ with sufficient module does not exceed 3. This together with (58) in Section 4 that the order p of pole of the resolvent operator of $\mathscr{A}$ is less than or equal to 3. It then follows from the general formula: $m_{(a\lambda)} \leqslant p \cdot m_{(g\lambda)} \leqslant 9$ (see e.g. [26, p. 148]), where $m_{(a\lambda)}$, $m_{(g\lambda)}$ denote the algebraic and geometric multiplicities, respectively (see the beginning of Section 4). Here, we used the fact from Lemma 3.1 that $m_{(g\lambda)} \leqslant 3$. The proof is complete. □

4. COMPLETENESS, RIESZ BASIS GENERATION AND STABILITY RESULT

Suppose $\mathbf{A}$ is a closed operator in a Hilbert space $\mathbf{H}$. $W \in D(\mathbf{A})$ is called a generalized eigenfunction of $\mathbf{A}$ associated with eigenvalue λ if there is an integer $n \geqslant 1$ such that $(\lambda - \mathbf{A})^n W = 0$. The root subspace of $\mathbf{A}$ which is denoted by $\mathrm{Sp}(\mathbf{A})$, is the closed subspace of $\mathbf{H}$ spanned by all generalized

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eigenfunctions of $\mathbf{A}$. Moreover, $m_{(a\lambda)} = \dim\{W|(\lambda - \mathbf{A})^n W = 0$ for some integer $n\}$ is called the algebraic multiplicity of λ. It is well known in functional analysis that each eigenvalue of a discrete operator must have finite algebraic multiplicity. A non-zero $W \in D(\mathbf{A})$ is called an eigenfunction of $\mathbf{A}$ corresponding to λ if $(\lambda - \mathbf{A})W = 0$. The number $m_{(g\lambda)} = \dim\{W|(\lambda - \mathbf{A})W = 0\}$ is called geometric multiplicity of λ.

Let $\mathscr{J}$ be a subset of integers. Recall that the sequence $\{W_i\}_{i\in\mathscr{J}}$ is called a basis for $\mathbf{H}$ if to each element $W \in \mathbf{H}$ corresponds a unique sequence of scalars $\{c_i\}$ such that the series

$$W = \sum_{i\in\mathscr{J}} c_i W_i \tag{48}$$

is convergent with respect to the norm of $\mathbf{H}$ (see [24, 27, 28]). $\{W_i\}_{i\in\mathscr{J}}$ is called a Riesz basis for $\mathbf{H}$ if

(a) $\overline{\text{span}}\{W_i\} = \mathbf{H}$;
(b) there exist some positive constants m_1 and m_2 such that for any numbers $c_i, i \in \mathscr{I}$, where $\mathscr{I}$ is any finite subset of $\mathscr{J}$, it has

$$m_1 \sum_{i\in\mathscr{I}} |c_i|^2 \leqslant \left\| \sum_{i\in\mathscr{I}} c_i W_i \right\|^2 \leqslant m_2 \sum_{i\in\mathscr{I}} |c_i|^2$$

A basis $\{W_i\}_{i\in\mathscr{J}}$ for $\mathscr{H}$ is called a Riesz basis with parentheses [29] if (48) converges in $\mathscr{H}$ after putting some of its terms in parentheses the arrangement of which does not depend on W. We refer to [30] for more details on Riesz basis.

In order to prove the completeness of the root subspace, we need the following Theorem 4.1 (see, e.g. [20]).

Theorem 4.1
Let $\mathbf{A}$ be the generator of a C_0-semigroup in a Hilbert space $\mathbf{H}$. Assume that $\mathbf{A}$ is a discrete operator and for $\lambda \in \rho(\mathbf{A})$, $R(\lambda, \mathbf{A})$ is of the form

$$R(\lambda, \mathbf{A})Y = \frac{G(\lambda)Y}{F_1(\lambda)} \quad \forall Y \in \mathbf{H}$$

where for each $Y \in \mathbf{H}$, $G(\lambda)Y$ is an $\mathbf{H}$-valued entire function with order less than or equal to ρ_1 and $F_1(\lambda)$ is a scalar entire function of order ρ_2. Let $\rho := \max\{\rho_1, \rho_2\} < \infty$ and an integer n so that $n - 1 \leqslant \rho < n$. If there are $n + 1$ rays γ_j, $j = 0, 1, 2, \ldots, n$, on the complex plane

$$\arg \gamma_0 = \frac{\pi}{2} < \arg \gamma_1 < \arg \gamma_2 < \cdots < \arg \gamma_n = \frac{3\pi}{2}$$

with

$$\arg \gamma_{j+1} - \arg \gamma_j \leqslant \frac{\pi}{n}, \quad 0 \leqslant j \leqslant n - 1$$

so that for any $Y \in \mathbf{H}$, $R(\lambda, \mathbf{A})Y$ is bounded on each ray γ_j, $0 < j < n$, as $|\lambda| \to \infty$, then $\text{Sp}(\mathbf{A}) = \mathbf{H}$.

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Theorem 4.2
Let $\mathscr{A}$ be defined by (11)–(12). Then the root subspaces of $\mathscr{A}$ is complete in $\mathscr{H}$, i.e. $\mathrm{Sp}(\mathscr{A}) = \mathscr{H}$.

Proof
Let $\lambda \in \rho(\mathscr{A})$. For any $F := [\varphi_1, \psi_1, \varphi_2, \psi_2, \varphi_3, \psi_3] \in \mathscr{H}$, let $Y = R(\lambda, \mathscr{A})F$. Then, $Y := [f_1, g_1, f_2, g_2, f_3, g_3] \in D(\mathscr{A})$ and

$$\begin{cases} \lambda f_1 - g_1 = \varphi_1, \quad \lambda g_1 + f_1^{(4)} = \psi_1, \quad f_1'''(0) = -kg_1(0) \\ \lambda f_2 - g_2 = \varphi_2, \quad \lambda g_2 + f_2^{(4)} = \psi_2, \quad f_2'''(0) = -kg_2(0) \\ \lambda f_3 - g_3 = \varphi_3, \quad \lambda g_3 + f_3^{(4)} = \psi_3, \quad f_3'''(0) = -kg_3(0) \\ f_1(1) = f_2(1) = f_3(1), \quad f_1'(1) = f_2'(1) = f_3'(1) \\ f_1''(0) = f_2''(0) = f_3''(0) = 0 \\ f_1''(1) + f_2''(1) + f_3''(1) = 0, \quad f_1'''(1) + f_2'''(1) + f_3'''(1) = 0 \end{cases}$$

Thus

$$\begin{cases} g_1 = \lambda f_1 - \varphi_1, \quad g_2 = \lambda f_2 - \varphi_2, \quad g_3 = \lambda f_3 - \varphi_3 \\ f_1^{(4)} + \lambda^2 f_1 = \psi_1 + \lambda\varphi_1, \quad f_2^{(4)} + \lambda^2 f_2 = \psi_2 + \lambda\varphi_2, \quad f_3^{(4)} + \lambda^2 f_3 = \psi_3 + \lambda\varphi_3 \\ f_1(1) = f_2(1) = f_3(1), \quad f_1'(1) = f_2'(1) = f_3'(1) \\ f_1''(0) = f_2''(0) = f_3''(0) = 0, \quad f_1''(1) + f_2''(1) + f_3''(1) = 0 \\ f_1'''(0) = -\lambda k f_1(0) + k\varphi_1(0), \quad f_2'''(0) = -\lambda k f_2(0) + k\varphi_2(0) \\ f_3'''(0) = -\lambda k f_3(0) + k\varphi_3(0), \quad f_1'''(1) + f_2'''(1) + f_3'''(1) = 0 \end{cases} \tag{49}$$

Using the transformation (27), (49) can be written into a compact form

$$\begin{cases} \Phi'(x) + A(\lambda)\Phi(x) = \Phi_1(x, \lambda) \\ W^0(\lambda)\Phi(0) + W^1(\lambda)\Phi(1) = \Phi_2 \end{cases} \tag{50}$$

where

$$\begin{cases} \Phi = [f_1, f_1', f_1'', f_1''', f_2, f_2', f_2'', f_2''', f_3, f_3', f_3'', f_3''']^\top \\ \Phi_1 := [0, 0, 0, \psi_1 + \lambda\varphi_1, 0, 0, 0, \psi_2 + \lambda\varphi_2, 0, 0, 0, \psi_3 + \lambda\varphi_3]^\top \\ \Phi_2 := [0, k\varphi_1(0), 0, k\varphi_2(0), 0, k\varphi_3(0), 0, 0, 0, 0, 0, 0]^\top \end{cases} \tag{51}$$

Let $\lambda = i\rho^2$ with $-\pi/2 < \arg\rho \leqslant \pi/2$. By Lemma 3.3, $\widehat{\Phi}(x, \rho)$ given by (37) is a fundamental matrix solution to (28). Hence (50) admits a unique solution of the following:

$$\Phi(x, \lambda) = \widehat{\Phi}(x, \rho)C + \int_0^x \widehat{\Phi}(x - \tau, \rho)P^{-1}(\rho)\Phi_1(\tau, \lambda)\,\mathrm{d}\tau \tag{52}$$

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where $P(\rho)$ is given by (32) and $C=[c_1, c_2, \ldots, c_{12}]^\top \in \mathbb{C}^{12}$ is a unique vector to be determined by the boundary conditions of (50). Substituting (52) into the second equation of (50), we obtain

$$W^0(\lambda)P(\rho)C + W^1(\lambda)\left[\widehat{\Phi}(1,\rho)C + \int_0^1 \widehat{\Phi}(1-\tau,\rho)P^{-1}(\rho)\Phi_1(\tau,\lambda)\,\mathrm{d}\tau\right] = \Phi_2$$

and hence

$$[W^0(\lambda)P(\rho) + W^1(\lambda)\widehat{\Phi}(1,\rho)]C = \Phi_2 - \int_0^1 W^1(\lambda)\widehat{\Phi}(1-\tau,\rho)P^{-1}(\rho)\Phi_1(\tau,\lambda)\,\mathrm{d}\tau$$

which is an algebraic equation

$$[T^R(\lambda)\widehat{\Phi}]C = \Phi_2 - \int_0^1 W^1(\lambda)\widehat{\Phi}(1-\tau,\rho)P^{-1}(\rho)\Phi_1(\tau,\lambda)\,\mathrm{d}\tau := \widetilde{C} \tag{53}$$

where $T^R(\lambda)\widehat{\Phi}$ is given by (39). Now we look for the entries of C. Note that

$$\widehat{\Phi}(x-\tau,\rho)P^{-1}(\rho)\Phi_1(\tau,\lambda) = \begin{bmatrix} U_1(x-\tau)\Phi_{11}(\tau,\lambda) \\ U_1(x-\tau)\Phi_{12}(\tau,\lambda) \\ U_1(x-\tau)\Phi_{13}(\tau,\lambda) \end{bmatrix} \tag{54}$$

where

$$\begin{cases} U_1(x) := P_1(\rho)E_1(x,\rho)P_1^{-1}(\rho) \\ \Phi_{1i}(x,\lambda) = [0,0,0,\psi_i(x)+\lambda\varphi_i(x)]^\top, \quad i=1,2,3 \end{cases} \tag{55}$$

and $E_1(x,\rho)$ is given by (36). A direct computation gives, for $i=1,2,3$

$$\begin{aligned} &U_1(x-\tau)\Phi_{1i}(\tau,\lambda) \\ &= \frac{\psi_i(\tau)+\lambda\varphi_i(\tau)}{4\rho^3}\begin{bmatrix} \mathrm{e}^{\rho(x-\tau)} & \mathrm{e}^{-\rho(x-\tau)} & \mathrm{e}^{\mathrm{i}\rho(x-\tau)} & \mathrm{e}^{-\mathrm{i}\rho(x-\tau)} \\ \rho\mathrm{e}^{\rho(x-\tau)} & -\rho\mathrm{e}^{-\rho(x-\tau)} & i\rho\mathrm{e}^{\mathrm{i}\rho(x-\tau)} & -i\rho\mathrm{e}^{-\mathrm{i}\rho(x-\tau)} \\ \rho^2\mathrm{e}^{\rho(x-\tau)} & \rho^2\mathrm{e}^{-\rho(x-\tau)} & -\rho^2\mathrm{e}^{\mathrm{i}\rho(x-\tau)} & -\rho^2\mathrm{e}^{-\mathrm{i}\rho(x-\tau)} \\ \rho^3\mathrm{e}^{\rho(x-\tau)} & -\rho^3\mathrm{e}^{-\rho(x-\tau)} & -i\rho^3\mathrm{e}^{\mathrm{i}\rho(x-\tau)} & i\rho^3\mathrm{e}^{-\mathrm{i}\rho(x-\tau)} \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ i \\ -i \end{bmatrix} \\ &= \frac{\psi_i(\tau)+\lambda\varphi_i(\tau)}{4\rho^3}\begin{bmatrix} \mathrm{e}^{\rho(x-\tau)} - \mathrm{e}^{-\rho(x-\tau)} + i\mathrm{e}^{\mathrm{i}\rho(x-\tau)} - i\mathrm{e}^{-\mathrm{i}\rho(x-\tau)} \\ \rho\mathrm{e}^{\rho(x-\tau)} + \rho\mathrm{e}^{-\rho(x-\tau)} - \rho\mathrm{e}^{\mathrm{i}\rho(x-\tau)} - \rho\mathrm{e}^{-\mathrm{i}\rho(x-\tau)} \\ \rho^2\mathrm{e}^{\rho(x-\tau)} - \rho^2\mathrm{e}^{-\rho(x-\tau)} - i\rho^2\mathrm{e}^{\mathrm{i}\rho(x-\tau)} + i\rho^2\mathrm{e}^{-\mathrm{i}\rho(x-\tau)} \\ \rho^3\mathrm{e}^{\rho(x-\tau)} + \rho^3\mathrm{e}^{-\rho(x-\tau)} + \rho^3\mathrm{e}^{\mathrm{i}\rho(x-\tau)} + \rho^3\mathrm{e}^{-\mathrm{i}\rho(x-\tau)} \end{bmatrix} \end{aligned}$$

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Substituting (54) into (52) and (53), we get the last term of (52) and $\widetilde{C}$. Hence, each entry of C is of the form:

$$c_i = \frac{\widetilde{\Delta}_i(\rho)}{\Delta(\rho)}, \quad i = 1, 2, \ldots, 12 \tag{56}$$

where $\Delta(\rho)$ is given by (38) and $\widetilde{\Delta}_i(\rho), i = 1, 2, \ldots, 12$, are the determinants of the matrices obtained by replacing the ith column of $T^R(\lambda)\widehat{\Phi}$ with $\widetilde{C}$. After a straightforward computation, we can get the unique solution $\Phi(x, \lambda)$ given by (52) to Equation (50). This in return gives the solutions to (49) by

$$f_1 = e_1^\top \Phi, \quad f_2 = e_5^\top \Phi, \quad f_3 = e_9^\top \Phi \tag{57}$$

where $e_i \in \mathbb{C}^{12}$ is the unit vector in which the ith entry is one and others are zeros. Hence, we conclude the unique solution $Y = R(\lambda, \mathscr{A})F$. Moreover, from (56), it is seen that $Y = R(\lambda, \mathscr{A})F$ can be written as

$$Y = R(\lambda, \mathscr{A})F = \frac{G(\lambda)Y}{\Delta(\lambda)}, \quad \lambda \in \rho(\mathscr{A}) \tag{58}$$

where $G(\lambda)Y$ is an $\mathscr{H}$-valued entire function of finite exponential type. Furthermore, on each of these rays $\gamma_0 = -M + i\mathbb{R}_+$, $\gamma_1 = -M - \mathbb{R}_+$ and $\gamma_0 = -M - i\mathbb{R}_+$ for sufficient large positive real number M, $R(\lambda, \mathscr{A})F$ is uniformly bounded. Hence all assumptions of Theorem 4.1 are fulfilled and therefore $\mathrm{Sp}(\mathscr{A}) = \mathscr{H}$.

Finally, we show the Riesz basis property for system (13), which is a more profound result for systems governed by partial differential equations. To do this, we need the following Theorem 4.3 (see, e.g. [16]).

Theorem 4.3
Let $\mathbf{H}$ be a separable Hilbert space, and $\mathbf{A}$ be the generator of a C_0-semigroup $T(t)$ on $\mathbf{H}$. Suppose the following conditions are fulfilled:

(1) $\sigma(\mathbf{A}) = \sigma_1(\mathbf{A}) \cup \sigma_2(\mathbf{A})$ and $\sigma_2(\mathbf{A}) = \{\lambda_k\}_{k=1}^{\infty}$ consists of isolated eigenvalues of finite algebraic multiplicity only;
(2) $\sup_{k\geqslant 1} m_{a\lambda_k} < \infty$;
(3) there is a constant α such that $\sup\{\mathrm{Re}\,\lambda | \lambda \in \sigma_1(\mathbf{A})\} \leqslant \alpha \leqslant \inf\{\mathrm{Re}\,\lambda | \lambda \in \sigma_2(\mathbf{A})\}$ and $\inf_{n\neq m} |\lambda_n - \lambda_m| > 0$.

Then the following assertions hold true:

(i) There exist two $T(t)$-invariant closed subspaces $\mathbf{H}_1$ and $\mathbf{H}_2$ such that $\sigma(\mathbf{A}|_{\mathbf{H}_1}) = \sigma_1(\mathbf{A})$, $\sigma(\mathbf{A}|_{\mathbf{H}_2}) = \sigma_2(\mathbf{A})$, and $\{E(\lambda_k, \mathbf{A})\mathbf{H}_2\}_{k=1}^{\infty}$ forms a Riesz basis for $\mathbf{H}_2$, i.e.
(a)

$$x_2 = \sum_{k=1}^{\infty} E(\lambda_k, \mathscr{A})x_2, \quad \forall x_2 \in X_2$$

where $E(\lambda_k, \mathbf{A})$ denotes the eigen-projection associated with λ_k;

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(b) there exist two positive constants C_1, C_2 independent of k and x_2 such that

$$C_1 \sum_{k=1}^{\infty} \|E(\lambda_k, \mathbf{A})x_2\|^2 \leqslant \left\| \sum_{k=1}^{\infty} E(\lambda_k, \mathbf{A})x_2 \right\|^2 \leqslant C_2 \sum_{k=1}^{\infty} \|E(\lambda_k, \mathbf{A})x_2\|^2$$

Furthermore,

$$\mathbf{H} = \overline{\mathbf{H}_1 \oplus \mathbf{H}_2}$$

(ii) if $\sup_{k\geqslant 1} \|E(\lambda_k, \mathbf{A})\| < \infty$, then $D(\mathbf{A}) \subset \mathbf{H}_1 \oplus \mathbf{H}_2 \subset \mathbf{H}$
(iii) $\mathbf{H}$ can decompose into the topological direct sum $\mathbf{H} = \mathbf{H}_1 \oplus \mathbf{H}_2$ if and only if

$$\sup_{n\geqslant 1} \left\| \sum_{k=1}^{n} E(\lambda_k, \mathbf{A}) \right\| < \infty.$$

Combining Theorems 4.2, 4.3 and 3.1, we get Theorem 4.4.

Theorem 4.4
System (13) is a Riesz spectral system (in the sense that there is a set of generalized eigenfunctions of $\mathscr{A}$, which forms a Riesz basis with parenthesis for $\mathscr{H}$, see [31]). Moreover, the spectrum-determined growth condition holds true, that is, $s(\mathscr{A}) = \omega(\mathscr{A})$, where $s(\mathscr{A}) := \sup\{\mathrm{Re}\,\lambda | \lambda \in \sigma(\mathscr{A})\}$ is the spectral bound of $\mathscr{A}$ and $\omega(\mathscr{A})$ is the growth bound of the semigroup $e^{\mathscr{A}t}$ generated by $\mathscr{A}$.

Proof
In view of Theorem 3.1, we may take $\sigma_2(\mathscr{A}) := \sigma(\mathscr{A})$ and $\sigma_1(\mathscr{A}) := \{-\infty\}$. Then conditions (1)–(3) of Theorem 4.3 with $\mathbf{A} := \mathscr{A}$, $\mathbf{H} := \mathscr{H}$ are satisfied. Moreover, Theorem 4.2 implies that $\mathbf{H}_1 = \{0\}$. Thus, by the first assertion of Theorem 4.3, we get that there is a sequence of the generalized eigenfunctions of $\mathscr{A}$, which forms a Riesz basis with parenthesis for $\mathscr{H}$. Finally, the spectrum-determined growth condition can be deduced directly from the Riesz basis generation and the uniform boundedness of algebraic multiplicities of eigenvalues (see [32]). The proof is complete.

Finally, as a consequence of Theorem 4.4, we state separately the stability result for system (13).

Theorem 4.5
The trajectories of system (13) converge exponentially to the zero eigenspace. Precisely, there exist constants $M, \omega > 0$ such that any mild solution $Y(t)$ [22] to equation (13) with initial value $Y_0 \in \mathscr{H}$ satisfies

$$\|Y(t) - \langle Y_0, \Psi_{01}^* \rangle \Phi_{01} - \langle Y_0, \Psi_{02}^* \rangle \Phi_{02} - \langle Y_0, \Psi_{03}^* \rangle \Phi_{03}\| \leqslant M e^{-\omega t} \|Y_0\|$$

where $\Phi_{01} := [1, 0, 1, 0, 1, 0]$, $\Phi_{02} := [x, 0, x, 0, x, 0]$ and $\Phi_{03} := [0, x, 0, x, 0, x]$ that are generalized eigenfunctions of $\mathscr{A}$ with respect to eigenvalue zero, and Ψ_{01}^*, Ψ_{02}^* and Ψ_{03}^*, in which they are three generalized eigenfunctions of $\mathscr{A}^*$, the adjoint of $\mathscr{A}$, with respect to eigenvalue zero, are bi-orthogonal to Φ_{01}, Φ_{02} and Φ_{03}, respectively, in $\mathscr{H}$.

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REFERENCES

1. Lagnese JE, Leugering G, Schmidt EJPG. *Modeling, Analysis and Control of Dynamic Elastic Multi-link Structures*. Systems & Control: Foundations & Applications. Birkhäuser: Boston, MA, 1994.
2. Rolewicz S. On controllability of systems of strings. *Studia Mathematica* 1970; **36**:105–110.
3. Chen G, Delfour C, Krall AM, Payre G. Modeling, stabilization and control of serially connected beams. *SIAM Journal on Control and Optimization* 1987; **25**:526–546.
4. Chen G, Krantz SG, Russell DL, Wayne CE, West HH, Coleman MP. Analysis, designs, and behavior of dissipative joints for coupled beams. *SIAM Journal on Applied Mathematics* 1989; **49**:1665–1693.
5. Lagnese JE, Leugering G, Schmidt EJPG. *Modelling and Controllability of Networks of Thin Beams, System Modelling and Optimization*. Lecture Notes in Control and Information Science, vol. 180. Springer: Berlin, 1992; 467–480.
6. Schmidt EJPG. On the modelling and exact controllability of networks of vibrating strings. *SIAM Journal on Control and Optimization* 1992; **30**:229–245.
7. Lagnese JE, Leugering G, Schmidt EJPG. On the analysis and control of hyperbolic systems associated with vibrating networks. *Proceedings of the Royal Society of Edinburgh Section A* 1994; **124**:77–104.
8. Dekoninck B, Nicaise S. Control of networks of Euler–Bernoulli beams. *ESAIM: Control, Optimization and Calculus of Variations* 1999; **4**:57–81.
9. Dekoninck B, Nicaise S. The eigenvalue problem for networks of beams. *Linear Algebra and its Applications* 2000; **314**:165–189.
10. Dáger R. Observation and control of vibrations in tree-shaped networks of strings. *SIAM Journal on Control and Optimization* 2004; **43**:590–623.
11. Guo BZ. Riesz basis approach to the stabilization of a flexible beam with a tip mass. *SIAM Journal on Control and Optimization* 2001; **39**:1736–1747.
12. Guo BZ, Wang JM, Yung SP. On C_0-semigroup generation and boundary stabilization of a flexible manipulator with rotatory inertia. *Differential and Integral Equations* 2005; **18**:1013–1038.
13. Guo BZ, Wang JM. The well-posedness and stability of a beam equation with conjugate variables appearing simultaneously at the same boundary. *IEEE Transactions on Automatic Control* 2005; **50**:2087–2093.
14. Guo BZ, Wang JM, Yung SP. On the C_0-semigroup generation and exponential stability resulting from a shear force feedback on a rotating beam. *Systems and Control Letters* 2005; **18**:1013–1038.
15. Wang JM, Guo BZ, Chentouf B. Boundary feedback stabilization of a three-layer sandwich beam: Riesz basis approach. *ESAIM: Control, Optimization and Calculus of Variations* 2006; **12**:12–34.
16. Wang JM, Xu GQ, Yung SP. Exponential stability for variable coefficients Rayleigh beams under boundary feedback control: a Riesz basis approach. *Systems and Control Letters* 2004; **51**:33–50.
17. Wang JM, Xu GQ, Yung SP. Exponential stabilization of laminated beams with structural damping and boundary feedback controls. *SIAM Journal on Control and Optimization* 2005; **44**(5):1575–1597.
18. Wang JM, Xu GQ, Yung SP. Riesz basis property, exponential stability of variable coefficient Euler–Bernoulli beam with indefinite dampings. *IMA Journal of Applied Mathematics* 2005; **70**(3):459–477.
19. Wang JM, Yung SP. Stability of a nonuniform Rayleigh beam with internal dampings. *Systems and Control Letters* 2006; **55**(10):863–870.
20. Xu GQ, Guo BZ. Riesz basis property of evolution equations in Hilbert spaces and application to a coupled string equation. *SIAM Journal on Control and Optimization* 2003; **42**:966–984.
21. Kato T. *Perturbation Theory of Linear Operators*. Springer: New York, 1976.
22. Pazy A. *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Springer: New York, 1983.
23. Birkhoff GD, Langer RE. The boundary problems and developments associated with a system of ordinary linear differential equations of the first order. *Proceedings of the American Academy of Arts and Sciences* 1923; **58**:49–128.
24. Tretter C. Spectral problems for systems of differential equations $y' + A_0 y = \lambda A_1 y$ with λ-polynomial boundary conditions. *Mathematische Nachrichten* 2000; **214**:129–172.
25. Tretter C. Boundary eigenvalue problems for differential equations $N\eta = \lambda P\eta$ with λ-polynomial boundary conditions. *Journal of Differential Equations* 2001; **170**:408–471.
26. Luo ZH, Guo BZ, Morgül Ö. *Stability and Stabilization of Linear Dimensional Systems with Applications*. Springer: London, 1999.
27. Gohberg IMG, Krein MG. *Introduction to Theory of Linear Nonselfadjoint Operators*. Transactions of Mathematics Monograph, vol. 18. AMS: Providence, RI, 1969.

28. Gohberg I, Goldberg S, Kaashoek MA. *Classes of Linear Operators*, vol. I. Birkhaüser: Basel, 1990.
29. Shkalikov AA. Boundary problems for ordinary differential equations with parameter in the boundary conditions. *Journal of Soviet Mathematics* 1986; **33**:1311–1342.
30. Young RM. *An Introduction to Nonharmonic Fourier Series*. Academic Press: London, 2001.
31. Curtain RF, Zwart HJ. *An Introduction to Infinite-dimensional Linear Systems Theory*. Springer: New York, 1995.
32. Guo BZ, Xu GQ. Expansion of solution in terms of generalized eigenfunctions for a hyperbolic system with static boundary condition. *Journal of Functional Analysis* 2006; **231**:245–268.

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