On the stability of swelling porous elastic soils with fluid saturation by one internal damping

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This article addressed the stabilization of a system of 1D swelling porous elastic soils with fluid saturation. The system is described by strongly coupled vibrating fluid and solid elastic materials. Using Riesz basis approach, we show that the whole system can be exponentially stabilized by only one internal viscous damping with variable feedback gain imposed in the fluid part, which is sharp contrast with the same effect by two dampings in existing literature. Moreover, the explicit asymptotic expressions of high eigenfrequencies exhibit clearly how this one damping can affect the another part of solid vibration.

Keywords: stabilization; swelling porous elastic soils; damping; spectral analysis; Riesz basis.

1. Introduction

It is generally recognized that the swelling of soils, plants, drying of fibres, wood, paper, etc belong to the porous media theory. A complete formulation of mixture theory for porous elastic solid filled with fluid and gas was developed by Eringen (1994). This formulation has many applications in various practical problems such as field of swelling, oil explanation, slurried and consolidation. One of the problems in this theory is on the interactions between two different components (see Wei & Muraleetharan, 2002). It was proved in Quintanilla (2002) that a 1D system can be exponentially stabilized by three internal dampings with constant feedback gains imposed in both solid and liquid equations.

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From the standpoint of system control theory, a very important and interesting question is naturally raising to see if one controller can achieve the stability for the system of mixed solid and fluid materials. There are plenty of studies for coupled infinite-dimensional systems on this aspect. Wang et al. (2006) adopted a boundary control strategy to stabilize a sandwich beam coupled with one auxiliary ordinary differential equation. Ammari et al. (2002) proposed two pointwise internal controls at joints to stabilize two connected beams and it was found that the exponential stability can not be achieved under one point control only if the proportion of the lengths of two beams is irrational (see also Guo & Chan, 2001), etc. The same occurs for weakly coupled vibrating systems. It was shown in Quintanilla (2003) that one internal viscous damping is not enough to stabilize exponentially the whole system of weakly coupled Timoshenko beams but two dampings are possible (see Shi & Feng, 2001).

In this article, we show that only one internal damping imposed in fluid equation is sufficient to stabilize exponentially the whole system of swelling porous elastic soils with fluid saturation. Same can also be obtained, via a same analysis in the present paper, for a damping only forced on another solid vibration. The advantage of this article in comparison with existing literature is that the feedback gain can be a function of spatial variable with minor requirement that this function should be positive in a measurable subset with positive measure in spatial space. The Riesz basis approach and matrix operator pencil method (see Adamjan et al., 2002) are adopted in investigation. Actually, our results are much more profound than stability itself. We show that (a) the generalized eigenfunctions of the system form a Riesz basis for the energy state space; (b) the asymptotics of eigenvalues are explicitly presented, which exhibit clearly how this one viscous damping affects quantitatively the whole system; (c) the spectrum-determined growth condition that is a difficult problem in partial differential equation system control is established and (d) the exponential stability is a direct consequence of results (a)–(c). To our best knowledge, this is a first attempt to stabilize a system of two coupled wave equations by only one internal damping.

We proceed as follows: In Section 2, we formulate the problem as an abstract evolution equation in the energy state space. Section 3 is devoted to the spectral analysis of the system, which is the main body of the article. The Riesz basis generation as well as exponential stability is presented in Section 4. Finally, we give some concluding remarks in Section 5.

2. Formulation of the problem

We consider a linear field equation of swelling porous elastic soils with fluid saturation of the following (Eringen, 1994; Quintanilla, 2002):

\begin{align}
\rho_z \frac{\partial^2 z}{\partial t^2}(x, t) &= a_1 \frac{\partial^2 z}{\partial x^2}(x, t) + a_2 \frac{\partial^2 u}{\partial x^2}(x, t) - \rho_z \gamma(x) \frac{\partial z}{\partial t}(x, t), \quad 0 < x < \ell, \quad t > 0,
\end{align}

\begin{align}
\rho_u \frac{\partial^2 u}{\partial t^2}(x, t) &= a_2 \frac{\partial^2 z}{\partial x^2}(x, t) + a_3 \frac{\partial^2 u}{\partial x^2}(x, t), \quad 0 < x < \ell, \quad t > 0,
\end{align}

where \(z(x, t)\) and \(u(x, t)\) represent the displacements of fluid and solid elastic materials at space position \(x \in (0, \ell)\) and time \(t > 0\), respectively. The constants \(\rho_z, \rho_u > 0\) are the densities of corresponding constituents, \(a_1, a_3 > 0\) are the constitutive constants, \(a_2 \neq 0\) can be either positive or negative, which demonstrates the strongly coupled physical property of the different materials described by (2.1)–(2.2), and \(\gamma\) is an internal viscous damping function.

The initial and boundary conditions of the system (2.1)–(2.2) are given by

\begin{align}
z(x, 0) &= z_0(x), \quad u(x, 0) = u_0(x), \quad x \in [0, \ell],
\end{align}
\[
\frac{d}{dt} z(x, 0) = z_1(x), \quad \frac{d}{dt} u(x, 0) = u_1(x), \quad x \in [0, \ell],
\]
(2.4)

\[
z(0, t) = \frac{d}{dx} z(\ell, t) = u(0, t) = \frac{d}{dx} u(\ell, t) = 0.
\]
(2.5)

The total energy for the system (2.1)–(2.5) is given by
\[
E(t) := \frac{1}{2} \int_0^\ell \left[ \rho_e |z_t(x, t)|^2 + \rho_a |u_t(x, t)|^2 + \left\langle \mathcal{Y} \left( \begin{array}{c} z_x(x, t) \\ u_x(x, t) \end{array} \right), \left( \begin{array}{c} z_t(x, t) \\ u_t(x, t) \end{array} \right) \right\rangle_{\mathbb{C}^2} \right] dx,
\]
(2.6)

where \( \langle \cdot, \cdot \rangle_{\mathbb{C}^2} \) denotes the inner product on \( \mathbb{C}^2 \) and the matrix
\[
\mathcal{Y} := \begin{pmatrix} a_1 & a_2 \\ a_2 & a_3 \end{pmatrix}
\]
(2.7)
is positive definite, i.e. \( a_1a_3 > a_2^2 \).

For simplicity, we assume \( \ell = 1 \) throughout the paper, otherwise, we can normalize \( \ell = 1 \) by change of variable \( x = \ell y \) with \( y \in (0, 1) \). We begin by formulating the system (2.1)–(2.5) into an abstract evolution equation in the state Hilbert space \( \mathcal{H} \):
\[
\mathcal{H} := (H^1_E(0, 1) \times L^2(0, 1))^2, \quad H^1_E(0, 1) := \{ f \in H^1(0, 1) \mid f(0) = 0 \},
\]
(2.8)

where \( H^1(0, 1) \) denotes the usual Sobolev space. Due to the energy function (2.6), it is natural to introduce the following inner product on \( \mathcal{H} \):
\[
\langle Y_1, Y_2 \rangle_{\mathcal{H}} := \int_0^1 \left[ \rho_e w_1 \bar{w}_2 + \rho_a v_1 \bar{v}_2 + \left\langle \mathcal{Y} \left( \begin{array}{c} z'_1 \\ u'_1 \end{array} \right), \left( \begin{array}{c} z'_2 \\ u'_2 \end{array} \right) \right\rangle_{\mathbb{C}^2} \right] dx,
\]
(2.9)

where \( Y_i := [z_i, w_i, u_i, v_i], \ i = 1, 2 \), and the prime “’” denotes the differentiation in \( x \). Define (unbounded) operator \( A \) by
\[
\mathcal{D}(A) := \{ [z, w, u, v] \in \mathcal{H} \mid z, u \in H^2(0, 1), w, v \in H^1_E(0, 1), z'(1) = u'(1) = 0 \},
\]
\[
A := \begin{bmatrix} z \\ w \\ u \\ v \end{bmatrix}^T := \begin{bmatrix} \frac{1}{\rho_e} (a_1 z'' + a_2 u'') \\ w \\ v \end{bmatrix}^T, \quad \forall \begin{bmatrix} z \\ w \\ u \\ v \end{bmatrix}^T \in \mathcal{D}(A),
\]
(2.10)

and bounded operator by
\[
B := \begin{bmatrix} z \\ w \\ u \\ v \end{bmatrix}^T := \begin{bmatrix} 0 \\ -\gamma w \\ 0 \\ 0 \end{bmatrix}^T, \quad \forall \begin{bmatrix} z \\ w \\ u \\ v \end{bmatrix}^T \in \mathcal{H}.
\]
(2.11)
Then the system (2.1)–(2.5) can be formulated into an abstract evolution equation in \( \mathcal{H} \):

\[
\begin{aligned}
\frac{d}{dt} Y(t) &= (A + B) Y(t), \quad t > 0, \quad \mathcal{D}(A + B) = \mathcal{D}(A), \\
Y(0) &= [z_0, z_1, u_0, u_1],
\end{aligned}
\tag{2.12}
\]

where \( Y(t) := [z(\cdot, t), z_t(\cdot, t), u(\cdot, t), u_t(\cdot, t)] \). The following three results are straightforward.

**Lemma 2.1** The operator \( A \) defined by (2.10) is skew-adjoint in \( \mathcal{H} \).

**Theorem 2.1** Let \( A \) and \( B \) be defined by (2.10) and (2.11), respectively. Then \( A \) and \( A + B \) are of compact resolvents and \( 0 \in \rho(A) \cap \rho(A + B) \). Therefore, the spectrum of \( A \) and \( A + B \) consists of isolated eigenvalues of finite algebraic multiplicity only.

**Proof.** We first prove that \( 0 \in \rho(A + B) \), which is equivalent to showing that \( (A + B)^{-1} \) exists and is everywhere defined. For any \( G := [g_1, g_2, g_3, g_4] \in \mathcal{H} \), we find a unique \( F := [f_1, f_2, f_3, f_4] \in \mathcal{D}(A + B) \) such that

\[
(A + B) F = G,
\]

i.e.

\[
\begin{aligned}
f_2(x) &= g_1(x), \quad f_4(x) = g_3(x), \\
a_1 f_1''(x) + a_2 f_2''(x) &= \rho_c(\gamma(x)) f_2(x) + g_2(x), \\
a_2 f_1''(x) + a_3 f_3''(x) &= \rho_u g_4(x).
\end{aligned}
\tag{2.13}
\tag{2.14}
\tag{2.15}
\]

In light of (2.13)–(2.15) and \( a_1 a_3 > a_2^2 \), it has

\[
\begin{aligned}
f_1''(x) &= \frac{a_3}{a_3 a_1 - a_2^2} c_1(x) - \frac{a_2}{a_3 a_1 - a_2^2} c_2(x), \\
f_3''(x) &= \frac{a_1}{a_3 a_1 - a_2^2} c_2(x) - \frac{a_2}{a_3 a_1 - a_2^2} c_1(x),
\end{aligned}
\tag{2.16}
\tag{2.17}
\]

where

\[
\begin{aligned}
c_1(x) &:= \rho_c(\gamma(x)) g_1(x) + g_2(x), \quad c_2(x) := \rho_u g_4(x).
\end{aligned}
\tag{2.18}
\]

By boundary conditions \( f_1(0) = f_1'(1) = f_3(0) = f_3'(1) = 0 \), it follows that

\[
\begin{aligned}
f_1(x) &= \frac{a_3}{a_3 a_1 - a_2^2} \int_0^x \int_1^\xi c_1(\zeta) d\zeta d\xi - \frac{a_2}{a_3 a_1 - a_2^2} \int_0^x \int_1^\xi c_2(\zeta) d\zeta d\xi, \\
f_3(x) &= \frac{a_1}{a_3 a_1 - a_2^2} \int_0^x \int_1^\xi c_2(\zeta) d\zeta d\xi - \frac{a_2}{a_3 a_1 - a_2^2} \int_0^x \int_1^\xi c_1(\zeta) d\zeta d\xi.
\end{aligned}
\tag{2.19}
\tag{2.20}
\]

Combining (2.13), (2.19) and (2.20), we obtain a unique solution \( F \) to \( (A + B) F = G \). Thus, \( (A + B)^{-1} \) exists and is everywhere defined. The compactness of \( (A + B)^{-1} \) then follows directly from the Sobolev’s embedding theorem.

If \( \gamma \equiv 0 \), then \( B = 0 \) and hence this completes the proof. \( \square \)
THEOREM 2.2 Let \( \mathcal{A} \) and \( \mathcal{B} \) be defined by (2.10) and (2.11), respectively. Then \( \mathcal{A} \) generates a \( C_0 \)-group on \( \mathcal{H} \) and so is \( \mathcal{A} + \mathcal{B} \) since \( \mathcal{B} \) is bounded.

**Proof.** Let \( \mathcal{B} = 0 \) (i.e. \( \gamma = 0 \)). Then Lemma 2.1 and Theorem 2.1 say that \( \mathcal{A} \) is skew-adjoint in \( \mathcal{H} \) and \( 0 \in \rho(\mathcal{A}) \). Thus, \( \mathcal{A} \) generates a \( C_0 \)-group on \( \mathcal{H} \) by the Stone theorem (see Theorem 10.8 of Pazy, 1983, pp. 41).

\( \square \)

3. Spectral analysis

This section is devoted to the spectral analysis for the system (2.1)–(2.5). Let \( \lambda \in \sigma(\mathcal{A} + \mathcal{B}) \) and \( Y_\lambda := [z, w, u, v] \) be an eigenfunction of \( \mathcal{A} + \mathcal{B} \) corresponding to \( \lambda \). Then \( (\mathcal{A} + \mathcal{B})Y_\lambda = \lambda Y_\lambda \) implies \( w = \lambda z, v = \lambda u \) and that \( z, u \) satisfy the following characteristic equations:

\[
\begin{aligned}
\rho_z \lambda^2 z(x) - a_1 z''(x) - a_2 u''(x) + \rho_z \gamma(x) \lambda z(x) &= 0, \quad 0 < x < 1, \\
\rho_u \lambda^2 u(x) - a_2 z''(x) - a_3 u''(x) &= 0, \quad 0 < x < 1,
\end{aligned}
\] (3.1)

with the boundary conditions

\[
z(0) = u(0) = z'(1) = u'(1) = 0.
\] (3.2)

For brevity in notation, we set

\[
r_1 := \sqrt{\frac{\rho_z}{a_1}}, \quad r_2 := \sqrt{\frac{\rho_u}{a_3}}, \quad a_4 := \frac{a_2}{a_1}, \quad a_5 := \frac{a_2}{a_3}, \quad \delta := \frac{1}{1 - a_4 a_5} = \frac{a_1 a_3}{a_1 a_3 - a_2^2} > 1.
\] (3.3)

Note that if \( a_2 = 0 \), there is no coupling for the system (2.1)–(2.2). So we always assume that \( a_2 \neq 0 \) in the sequel. Now (3.1) becomes

\[
\begin{aligned}
r_1^2 \lambda^2 z(x) - z''(x) - a_4 u''(x) + r_1^2 \gamma(x) \lambda z(x) &= 0, \quad 0 < x < 1, \\
r_2^2 \lambda^2 u(x) - a_5 z''(x) - u''(x) &= 0, \quad 0 < x < 1,
\end{aligned}
\] (3.4)

which is equivalent to the following:

\[
\begin{aligned}
r_1^2 \lambda^2 z(x) - a_4 r_2^2 \lambda^2 u(x) - (1/\delta) z''(x) + r_1^2 \gamma(x) \lambda z(x) &= 0, \quad 0 < x < 1, \\
r_2^2 \lambda^2 u(x) - a_5 r_1^2 \lambda^2 z(x) - (1/\delta) u''(x) - a_5 r_1^2 \gamma(x) \lambda z(x) &= 0, \quad 0 < x < 1.
\end{aligned}
\] (3.5)

Set

\[
\Phi(x) := [z_1, z_2, u_1, u_2]^\top = [z, z', u, u']^\top.
\] (3.6)

Then (3.5) becomes

\[
T^D(x, \lambda) \Phi(x) = 0,
\] (3.7)

where

\[
T^D(x, \lambda) \Phi(x) := \Phi'(x) + A(x, \lambda) \Phi(x), \quad A(x, \lambda) := A_0 - \lambda A_1(x) - \lambda^2 A_2,
\] (3.8)
with $A_0$, $A_1$ and $A_2$ given by

\[
A_0 := \begin{bmatrix}
0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0
\end{bmatrix}, \quad A_1(x) := \begin{bmatrix}
0 & 0 & 0 & 0 \\
\delta r_1^2 \gamma(x) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-\alpha_5 \delta r_1^2 \gamma(x) & 0 & 0 & 0
\end{bmatrix}, \quad (3.9)
\]

\[
A_2 := \begin{bmatrix}
0 & 0 & 0 & 0 \\
\delta r_1^2 & 0 & -\alpha_4 \delta r_2^2 & 0 \\
0 & 0 & 0 & 0 \\
-\alpha_5 \delta r_1^2 & 0 & \delta r_2^2 & 0
\end{bmatrix}. \quad (3.10)
\]

Under the same formulation, the boundary condition (3.2) becomes

\[
T^R \Phi(x) := W^0 \Phi(0) + W^1 \Phi(1) = 0, \quad (3.11)
\]

where

\[
W^0 := \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}, \quad W^1 := \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}. \quad (3.12)
\]

Summarizing, we have proved the following Theorem 3.1.

**THEOREM 3.1** The characteristic equations (3.1), (3.2) are equivalent to the first-order linear system (3.7), (3.11). Moreover, $\lambda \in \sigma(A + B)$ if and only if (3.7), (3.11) possess a non-zero solution.

Next we utilize a standard technique due to Birkhoff & Langer (1923) and Tretter (2000, 2001) to expand the characteristic determinant of (3.7), (3.11). To begin with, we diagonalize the leading term $\lambda^2 A_2$ in (3.8). Let

\[
r_3 := \sqrt{\frac{\delta (r_1^2 + r_2^2) + \sqrt{\delta^2 (r_1^2 - r_2^2)^2 + (\delta^2 - \delta) r_1^2 r_2^2}}{2}} > 0, \quad (3.13)
\]

\[
r_4 := \sqrt{\frac{\delta (r_1^2 + r_2^2) - \sqrt{\delta^2 (r_1^2 - r_2^2)^2 + (\delta^2 - \delta) r_1^2 r_2^2}}{2}} > 0, \quad (3.14)
\]

\[
s := -\frac{\delta a_4 r_2^2}{r_3^2 - \delta r_1^2} = -\frac{r_3^2 - \delta r_2^2}{\delta a_5 r_1^2}, \quad t := -\frac{\delta a_4 r_2^2}{r_4^2 - \delta r_2^2} = -\frac{r_4^2 - \delta r_2^2}{\delta a_5 r_1^2}, \quad \delta_1 := \frac{1}{s - t}. \quad (3.15)
\]

It is easily checked that the above constants have the following relations that will be used later:

\[
r_3^2 + r_4^2 = \delta r_1^2 + \delta r_2^2, \quad s \delta_1 > 0, \quad t \delta_1 < 0, \quad (3.16)
\]

\[
r_3 \neq r_4, \quad 1 + ta_5 = \frac{r_2^2}{\delta r_1^2} > 0, \quad 1 + sa_5 = \frac{r_4^2}{\delta r_1^2} > 0. \quad (3.17)
\]
Define an invertible matrix $P(\lambda)$ in $\lambda \in \mathbb{C}$ of the form

$$P(\lambda) := S \begin{bmatrix} P_1(\lambda) & 0 \\ 0 & P_2(\lambda) \end{bmatrix}, \quad P(\lambda)^{-1} := \begin{bmatrix} P_1^{-1}(\lambda) & 0 \\ 0 & P_2^{-1}(\lambda) \end{bmatrix} S^{-1}, \quad \lambda \in \mathbb{C}, \; \lambda \neq 0, \quad (3.18)$$

where

$$P_1(\lambda) := \begin{bmatrix} r_3 \lambda & \lambda \gamma(x) \\ r_3 \gamma(x) & -r_3 \lambda \end{bmatrix}, \quad P_1^{-1}(\lambda) := \begin{bmatrix} \frac{1}{2r_3 \lambda} & \frac{1}{2r_3^2 \lambda} \\ \frac{1}{2r_3 \lambda} & \frac{-1}{2r_3^2 \lambda} \end{bmatrix},$$

$$P_2(\lambda) := \begin{bmatrix} r_4 \lambda & \lambda \gamma(x) \\ r_4 \gamma(x) & -r_4 \lambda \end{bmatrix}, \quad P_2^{-1}(\lambda) := \begin{bmatrix} \frac{1}{2r_4 \lambda} & \frac{1}{2r_4^2 \lambda} \\ \frac{1}{2r_4 \lambda} & \frac{-1}{2r_4^2 \lambda} \end{bmatrix},$$

$$S := \begin{bmatrix} sI_2 & tI_2 \\ I_2 & I_2 \end{bmatrix}, \quad S^{-1} := \delta \begin{bmatrix} I_2 & -tI_2 \\ -I_2 & sI_2 \end{bmatrix},$$

in which $I_2$ is a $2 \times 2$ identity matrix. It is seen that $P(\lambda)$ is a polynomial of degree 2 in $\lambda$. Letting

$$\Psi(x) := P^{-1}(\lambda) \Phi(x), \quad \tilde{T}^D(x, \lambda) := P(\lambda)^{-1} T^D(x, \lambda) P(\lambda),$$

we have

$$\tilde{T}^D(x, \lambda) \Psi(x) = \Psi'(x) + \tilde{A}(x, \lambda) \Psi(x) = 0, \quad \tilde{A}(x, \lambda) := P(\lambda)^{-1} A(x, \lambda) P(\lambda).$$

Since

$$S^{-1} A(x, \lambda) S = S^{-1} \begin{bmatrix} 0 & -1 & 0 & 0 \\ -\lambda^2 \delta a_1 r_1^2 & -\lambda \delta a_1 r_1^2 \gamma(x) & 0 & \lambda^2 \delta a_2 r_2^2 \\ 0 & 0 & 0 & -1 \\ \lambda^2 \delta a_5 r_1^2 + \lambda a_5 \delta r_1^2 \gamma(x) & 0 & -\lambda^2 \delta r_2^2 & 0 \end{bmatrix} S$$

$$= S^{-1} \begin{bmatrix} 0 & -s & 0 & -t^T \\ -s \lambda \delta a_1 r_1^2 \gamma(x) - \lambda^2 s r_3^2 & 0 & -t \lambda \delta a_1 r_1^2 \gamma(x) - \lambda^2 t r_3^2 & 0 \\ 0 & -1 & 0 & -1 \\ s \lambda \delta a_5 r_1^2 \gamma(x) - \lambda^2 t r_3^2 & 0 & t \lambda \delta a_5 r_1^2 \gamma(x) - \lambda^2 t r_3^2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & 0 & 0 \\ -\lambda^2 r_3^2 + \lambda \gamma_3(x) & 0 & \lambda \gamma_4(x) & 0 \\ 0 & 0 & 0 & -1 \\ \lambda \gamma_5(x) & 0 & -\lambda^2 r_4^2 + \lambda \gamma_6(x) & 0 \end{bmatrix}.$$
where

\[
\begin{align*}
\gamma_3(x) & := -s\tilde{A}_1 r_1^2 (1 + t\alpha) \gamma(x), \\
\gamma_4(x) & := -t\tilde{A}_1 r_1^2 (1 + t\alpha) \gamma(x), \\
\gamma_5(x) & := s\tilde{A}_1 r_1^2 (1 + s\alpha) \gamma(x), \\
\gamma_6(x) & := t\tilde{A}_1 r_1^2 (1 + s\alpha) \gamma(x),
\end{align*}
\]

(3.24)

we have

\[
\tilde{A}(x, \lambda) = \begin{bmatrix}
-\frac{1}{2} + \frac{\gamma_1(x)}{2r_2^2 \lambda} & -\frac{1}{2} & -\frac{\gamma_4(x)}{2r_2^2 \lambda} & 0 \\
\frac{1}{2} - \frac{\gamma_1(x)}{2r_2^2 \lambda} & -\frac{1}{2} & -\frac{\gamma_4(x)}{2r_2^2 \lambda} & 0 \\
\frac{\gamma_5(x)}{2r_2^2 \lambda} & 0 & -\frac{1}{2} + \frac{\gamma_6(x)}{2r_2^2 \lambda} & -\frac{1}{2} \\
-\frac{\gamma_5(x)}{2r_2^2 \lambda} & 0 & -\frac{1}{2} - \frac{\gamma_6(x)}{2r_2^2 \lambda} & -\frac{1}{2} \\
\end{bmatrix}
\]

(3.25)

\[
= -\lambda \hat{A}_1 - \hat{A}_0(x),
\]

where

\[
\gamma_7(x) := \frac{\gamma_3(x)}{2r_3}, \quad \gamma_8(x) := \frac{r_4 \gamma_4(x)}{2r_3^2}, \quad \gamma_9(x) := \frac{r_3 \gamma_5(x)}{2r_3^2}, \quad \gamma_{10}(x) := \frac{\gamma_6(x)}{2r_4},
\]

(3.26)

\[
\hat{A}_1 := \text{diag}(r_3, -r_3, r_4, -r_4), \quad \hat{A}_0(\cdot) := \begin{bmatrix}
-\gamma_7(\cdot) & -\gamma_7(\cdot) & -\gamma_8(\cdot) & -\gamma_8(\cdot) \\
\gamma_7(\cdot) & \gamma_7(\cdot) & \gamma_8(\cdot) & \gamma_8(\cdot) \\
-\gamma_9(\cdot) & -\gamma_9(\cdot) & -\gamma_{10}(\cdot) & -\gamma_{10}(\cdot) \\
\gamma_9(\cdot) & \gamma_9(\cdot) & \gamma_{10}(\cdot) & \gamma_{10}(\cdot)
\end{bmatrix}.
\]

(3.27)

**Theorem 3.2** Let 0 \( \neq \lambda \in \mathbb{C} \), and let \( r_3, r_4 \) be defined by (3.24). For \( x \in [0, 1] \), set

\[
E(x, \lambda) := \text{diag}(e^{r_3 \lambda x}, e^{-r_3 \lambda x}, e^{r_4 \lambda x}, e^{-r_4 \lambda x}).
\]

(3.28)

Then there exists a fundamental matrix solution \( \hat{\Psi}(x, \lambda) \) to the system (3.23), such that for large enough \( |\lambda| \),

\[
\hat{\Psi}(x, \lambda) = (\hat{\Psi}_0(x) + O(\lambda^{-1})) E(x, \lambda),
\]

(3.29)

where

\[
\hat{\Psi}_0(x) := \text{diag} \left( e^{-\int_0^x \gamma_7(\xi) d\xi}, e^{\int_0^x \gamma_7(\xi) d\xi}, e^{-\int_0^x \gamma_{10}(\xi) d\xi}, e^{\int_0^x \gamma_{10}(\xi) d\xi} \right).
\]

(3.30)
Proof. By (3.23), (3.25), the Assumption 2.1 of Tretter (2000, p. 134) is satisfied and hence a straightforward application of Theorem 2.2 of Tretter (2000, p. 135) (see also Birkhoff & Langer, 1923) shows that a fundamental matrix solution of (3.23) is of the following form:

$$\hat{\Psi}(x, \lambda) = (\hat{\Psi}_0(x) + \lambda^{-1}\hat{\Psi}_1(x) + \lambda^{-2}\hat{\Theta}(x, \lambda))E(x, \lambda),$$

where $\hat{\Theta}(x, \lambda)$ is uniformly bounded in $\lambda$ and $x \in [0, 1]$. Since $\hat{A}_1$ given by (3.27) is a diagonal matrix, it follows that $E(x, \lambda)$ given by (3.28) is a fundamental matrix solution to (3.23) involving only the leading order terms, in other words,

$$E'(x, \lambda) = \lambda \hat{A}_1 E(x, \lambda).$$

Next, compute $\hat{\Psi}'(x, \lambda)$ and $-\hat{\lambda}(x, \lambda) \hat{\Psi}(x, \lambda)$ to yield

$$\hat{\Psi}'(x, \lambda) = (\hat{\Psi}_0'(x) + \lambda^{-1}\hat{\Psi}_1'(x) + \lambda^{-2}\hat{\Theta}_x(x, \lambda))E(x, \lambda)$$

$$+ \lambda(\hat{\Psi}_0(x) + \lambda^{-1}\hat{\Psi}_1(x) + \lambda^{-2}\hat{\Theta}(x, \lambda))\hat{A}_1 E(x, \lambda),$$

$$-\hat{\lambda}(x, \lambda) \hat{\Psi}(x, \lambda) = (\lambda \hat{A}_1 + \hat{A}_0(x))(\hat{\Psi}_0(x) + \lambda^{-1}\hat{\Psi}_1(x) + \lambda^{-2}\hat{\Theta}(x, \lambda))E(x, \lambda).$$

Insert the above two equations into (3.23) and equate the corresponding coefficients of $\lambda^i$, $i = 1, 0, -1$, to give

$$\hat{\Psi}_0(x)\hat{A}_1 - \hat{A}_1 \hat{\Psi}_0(x) = 0, \quad (3.31)$$

$$\hat{\Psi}_0'(x) - \hat{A}_0(x)\hat{\Psi}_0(x) + \hat{\Psi}_1(x)\hat{A}_1 - \hat{A}_1 \hat{\Psi}_1(x) = 0. \quad (3.32)$$

It remains to show that the leading order term $\hat{\Psi}_0(x)$ is given by (3.30). In fact, from (3.31) and $r_3 \neq r_4$ in (17), we see that the matrix function $\hat{\Psi}_0(x)$ is of diagonal

$$\hat{\Psi}_0(x) := \text{diag}[\psi_{11}(x), \psi_{22}(x), \psi_{33}(x), \psi_{44}(x)]$$

and its entries can be obtained by substituting them into (3.32) to get

$$\begin{cases}
\psi_{11}' = -\gamma_7(x)\psi_{11}, & \psi_{22}' = \gamma_7(x)\psi_{22}, \\
\psi_{33}' = -\gamma_{10}(x)\psi_{33}, & \psi_{44}' = \gamma_{10}(x)\psi_{44},
\end{cases} \quad (3.33)$$

with $\hat{\Psi}_0(0) = I$. Equation (3.30) follows. \hfill \Box

Corollary 3.1 Let $\hat{\Phi}(x, \lambda)$ given by (3.29) be a fundamental matrix solution to system (3.23). Then

$$\hat{\Phi}(x, \lambda) := P(\lambda)\hat{\Psi}(x, \lambda) \quad (3.34)$$

is a fundamental matrix solution to the first-order linear system (3.7).

We are now in a position to estimate the asymptotics of the eigenvalues. Note that the eigenvalues of the first-order linear system (3.7), (3.11) are given by the zeros of the characteristic determinant

$$\Delta(\lambda) := \text{det}(T^R \hat{\Phi}(x, \lambda)), \quad \lambda \in \mathbb{C}, \quad (3.35)$$
where the operator $T^R$ is given by (3.11) and $\tilde{\Phi}(x, \lambda)$ is any fundamental matrix to the equation $T^D(x, \lambda)\Phi(x) = 0$ (see Tretter, 2000). By substituting (3.34), (3.29) and (3.30) into (3.35) and taking the boundary conditions (3.12) into account, we can obtain the asymptotic expressions for the eigenvalues. Actually, since

$$T^R \tilde{\Phi}(x, \lambda) = W^0 P(\lambda) \tilde{\Psi}(0, \lambda) + W^1 P(\lambda) \tilde{\Psi}(1, \lambda),$$

(3.36)
a simple computation by using (3.12) and (3.18) gives

$$W^0 P(\lambda) = W^0 \begin{bmatrix} \begin{smallmatrix} s P_1(\lambda) & t P_2(\lambda) \\ P_1(\lambda) & P_2(\lambda) \end{smallmatrix} \end{bmatrix} = \begin{bmatrix} s r_3 \lambda & s r_3 \lambda & tr_4 \lambda & tr_4 \lambda \\ r_3 \lambda & r_3 \lambda & r_4 \lambda & r_4 \lambda \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$W^1 P(\lambda) = W^1 \begin{bmatrix} \begin{smallmatrix} s P_1(\lambda) & t P_2(\lambda) \\ P_1(\lambda) & P_2(\lambda) \end{smallmatrix} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ sr_3^2 \lambda^2 & -sr_3^2 \lambda^2 & tr_4^2 \lambda^2 & -tr_4^2 \lambda^2 \\ r_3^2 \lambda^2 & -r_3^2 \lambda^2 & r_4^2 \lambda^2 & -r_4^2 \lambda^2 \end{bmatrix}.$$
Then

\[
T^R \Phi(x, \lambda) = \begin{bmatrix}
    sr_3 \lambda [1]_1 & sr_3 \lambda [1]_1 & tr_4 \lambda [1]_1 & tr_3 \lambda [1]_1 \\
    r_4 \lambda [1]_1 & r_3 \lambda [1]_1 & r_4 \lambda [1]_1 & r_3 \lambda [1]_1 \\
    sr_3^2 \lambda^2 [1]_1 e^{r_3 \lambda - f_0^1 \gamma_1(\xi) d\xi} & -sr_3^2 \lambda^2 [1]_1 e^{-r_3 \lambda + f_0^1 \gamma_1(\xi) d\xi} & tr_4^2 \lambda^2 [1]_1 e^{r_4 \lambda - f_0^1 \gamma_1(\xi) d\xi} & -tr_4^2 \lambda^2 [1]_1 e^{-r_4 \lambda + f_0^1 \gamma_1(\xi) d\xi} \\
    r_3^2 \lambda^2 [1]_1 e^{r_3 \lambda - f_0^1 \gamma_1(\xi) d\xi} & -r_3^2 \lambda^2 [1]_1 e^{-r_3 \lambda + f_0^1 \gamma_1(\xi) d\xi} & r_4^2 \lambda^2 [1]_1 e^{r_4 \lambda - f_0^1 \gamma_1(\xi) d\xi} & -r_4^2 \lambda^2 [1]_1 e^{-r_4 \lambda + f_0^1 \gamma_1(\xi) d\xi} \\
\end{bmatrix}.
\]

and hence

\[
\Delta(\lambda) = \text{det}(T^R \Phi(x, \lambda)) = \lambda^6 \left\{ \text{det} \begin{bmatrix}
    0 & 0 & (t - s)r_4 & (t - s)r_4 \\
    r_3 & r_3 & r_4 & r_4 \\
    0 & 0 & (t - s)r_4 e^{r_3 \lambda - f_0^1 \gamma_1(\xi) d\xi} & (s - t) r_4 e^{-r_4 \lambda + f_0^1 \gamma_1(\xi) d\xi} \\
    r_3^2 e^{r_3 \lambda - f_0^1 \gamma_1(\xi) d\xi} & -r_3^2 e^{-r_3 \lambda + f_0^1 \gamma_1(\xi) d\xi} & r_4^2 e^{r_4 \lambda - f_0^1 \gamma_1(\xi) d\xi} & -r_4^2 e^{-r_4 \lambda + f_0^1 \gamma_1(\xi) d\xi} \\
\end{bmatrix} \right\} + O(\lambda^{-1})
\]

\[
= -\lambda^6 \left\{ \text{det} \begin{bmatrix}
    r_3 & r_3 \\
    r_3^2 e^{r_3 \lambda - f_0^1 \gamma_1(\xi) d\xi} & -r_3^2 e^{-r_3 \lambda + f_0^1 \gamma_1(\xi) d\xi} \\
\end{bmatrix} \right\} + O(\lambda^{-1})
\]

\[
= -r_3^3 r_4^3 (t - s)^2 \lambda^6 \left\{ \text{det} \begin{bmatrix}
    1 & 1 \\
    e^{r_3 \lambda - f_0^1 \gamma_1(\xi) d\xi} & -e^{-r_3 \lambda + f_0^1 \gamma_1(\xi) d\xi} \\
\end{bmatrix} \right\} + O(\lambda^{-1})
\]

\[
= -r_3^3 r_4^3 (t - s)^2 \lambda^6 \left\{ \left( e^{r_3 \lambda - f_0^1 \gamma_1(\xi) d\xi} + e^{-r_3 \lambda + f_0^1 \gamma_1(\xi) d\xi} \right) \right\} + O(\lambda^{-1})
\]

\[
= -r_3^3 r_4^3 (t - s)^2 \lambda^6 \left\{ \left( e^{r_3 \lambda - f_0^1 \gamma_1(\xi) d\xi} + e^{-r_3 \lambda + f_0^1 \gamma_1(\xi) d\xi} \right) \right\} + O(\lambda^{-1})
\]

Theorem 3.3 Let \( \Delta(\lambda) \) be the characteristic determinant of the first-order linear systems (3.7), (3.11). Then \( \Delta(\lambda) \) has the following asymptotic expansion:

\[
\Delta(\lambda) = \mu \lambda^6 \{ \Delta_1 \times \Delta_2 + O(\lambda^{-1}) \} \quad \text{as} \ |\lambda| \to \infty,
\]

(3.37)
where $\mu := -r_3^3 r_4^3 (t - s)^2$,

$$\Delta_1 := e^{r_3 \lambda - \int_0^1 \gamma_7(\xi) d\xi} + e^{-r_3 \lambda + \int_0^1 \gamma_7(\xi) d\xi},$$  \hspace{1cm} (3.38)

$$\Delta_2 := e^{r_4 \lambda - \int_0^1 \gamma_{10}(\xi) d\xi} + e^{-r_4 \lambda + \int_0^1 \gamma_{10}(\xi) d\xi}.$$  \hspace{1cm} (3.39)

Now the characteristic determinant $\Delta(\lambda) = 0$ becomes

$$\Delta_1 \times \Delta_2 + O(\lambda^{-1}) = 0,$$

which is equivalent to

$$e^{r_3 \lambda - \int_0^1 \gamma_7(\xi) d\xi} + e^{-r_3 \lambda + \int_0^1 \gamma_7(\xi) d\xi} + O(\lambda^{-1}) = 0$$  \hspace{1cm} (3.40)

or

$$e^{r_4 \lambda - \int_0^1 \gamma_{10}(\xi) d\xi} + e^{-r_4 \lambda + \int_0^1 \gamma_{10}(\xi) d\xi} + O(\lambda^{-1}) = 0.$$  \hspace{1cm} (3.41)

By the Rouché’s theorem, the roots of (3.40) can be estimated by those of

$$e^{r_3 \lambda - \int_0^1 \gamma_7(\xi) d\xi} + e^{-r_3 \lambda + \int_0^1 \gamma_7(\xi) d\xi} = 0$$

that can be found explicitly as following

$$\tilde{\lambda}_{1k} = \frac{1}{r_3} \left( \int_0^1 \gamma_7(\xi) d\xi + \left( k + \frac{1}{2} \right) \pi i \right), \hspace{1cm} k \in \mathbb{Z},$$  \hspace{1cm} (3.42)

where $\gamma_7(x)$ is defined by (3.26). Thus, the roots of (3.40) satisfy

$$\lambda_{1k} = \frac{1}{r_3} \left( \int_0^1 \gamma_7(\xi) d\xi + \left( k + \frac{1}{2} \right) \pi i \right) + O(k^{-1}), \hspace{1cm} |k| \geq N_1, \hspace{0.5cm} k \in \mathbb{Z},$$  \hspace{1cm} (3.43)

where $N_1$ is some sufficiently large positive integer. Repeating the process for (3.41), we get the asymptotics of the second branch of eigenvalues:

$$\lambda_{2k} = \frac{1}{r_4} \left( \int_0^1 \gamma_{10}(\xi) d\xi + \left( k + \frac{1}{2} \right) \pi i \right) + O(k^{-1}), \hspace{1cm} |k| \geq N_2, \hspace{0.5cm} k \in \mathbb{Z}$$  \hspace{1cm} (3.44)

for some sufficiently large positive integer $N_2$.

Eventually, we have proved the following Theorem 3.4 for the spectrum of $\mathcal{A} + \mathcal{B}$.

**Theorem 3.4** Let $\mathcal{A} + \mathcal{B}$ be defined in (2.12). Then each $\lambda \in \sigma(\mathcal{A} + \mathcal{B})$ is algebraically simple when $|\lambda|$ is large enough, and the following asymptotic expressions hold:

$$\lambda_{1k} = \frac{1}{r_3} \left( \int_0^1 \gamma_7(\xi) d\xi + \left( k + \frac{1}{2} \right) \pi i \right) + O(k^{-1}),$$  \hspace{1cm} (3.45)

$$\lambda_{2k} = \frac{1}{r_4} \left( \int_0^1 \gamma_{10}(\xi) d\xi + \left( k + \frac{1}{2} \right) \pi i \right) + O(k^{-1}).$$  \hspace{1cm} (3.46)
for $|k| \geq \max\{N_1, N_2\}, k \in \mathbb{Z}$, where $N_1, N_2$ are large enough positive integers, $r_3, r_4 > 0$ depend on system parameters only and $\gamma_7, \gamma_{10}$ are given by (3.3), (3.13)–(3.17), (3.24) and (3.26). A simple computation shows that

$$\frac{1}{r_3} \int_0^1 \gamma_7(\xi)d\xi = -\frac{1}{4} \frac{\delta(r_1^2 - r_2^2) + \sqrt{\delta^2(r_1^2 - r_2^2)^2 + (\delta^2 - \delta)r_1^2 r_2^2}}{\sqrt{\delta^2(r_1^2 - r_2^2)^2 + (\delta^2 - \delta)r_1^2 r_2^2}} \int_0^1 \gamma(\xi)d\xi$$

$$\frac{1}{r_4} \int_0^1 \gamma_7(\xi)d\xi = -\frac{1}{4} \frac{\delta(r_2^2 - r_1^2) + \sqrt{\delta^2(r_2^2 - r_1^2)^2 + (\delta^2 - \delta)r_2^2 r_1^2}}{\sqrt{\delta^2(r_2^2 - r_1^2)^2 + (\delta^2 - \delta)r_2^2 r_1^2}} \int_0^1 \gamma(\xi)d\xi,$$

where

$$r_1 = \frac{\rho_z}{\sqrt{a_1}}, \quad r_2 = \frac{\rho_a}{\sqrt{a_3}}, \quad \delta = \frac{a_1 a_3}{a_1 a_3 - a_2^2} > 1.$$ 

Therefore

$$\text{Re}\lambda_{1k} \to -\frac{1}{4} \frac{\delta(r_1^2 - r_2^2) + \sqrt{\delta^2(r_1^2 - r_2^2)^2 + (\delta^2 - \delta)r_1^2 r_2^2}}{\sqrt{\delta^2(r_1^2 - r_2^2)^2 + (\delta^2 - \delta)r_1^2 r_2^2}} \int_0^1 \gamma(\xi)d\xi \quad \text{as } k \to \infty,$$  

$$\text{Re}\lambda_{2k} \to -\frac{1}{4} \frac{\delta(r_2^2 - r_1^2) + \sqrt{\delta^2(r_2^2 - r_1^2)^2 + (\delta^2 - \delta)r_2^2 r_1^2}}{\sqrt{\delta^2(r_2^2 - r_1^2)^2 + (\delta^2 - \delta)r_2^2 r_1^2}} \int_0^1 \gamma(\xi)d\xi \quad \text{as } k \to \infty.$$ 

**Remark 3.1** From Theorem 3.4, it is found that

$$\text{Re}\lambda_{1k} + \text{Re}\lambda_{2k} \to -\frac{1}{2} \int_0^1 \gamma(\xi)d\xi \quad \text{as } k \to \infty,$$

which exhibits clearly how the damping function $\gamma$ affects simultaneously the two coupled wave equations.

### 4. Exponential stability

In this section, we examine the stability property for the system (2.12). The following classic Keldysh’s theorem (see Theorem 4.1 of Gohberg et al., 1990, pp. 170) will be applied to get the completeness of the generalized eigenfunctions of $A + B$.

**Lemma 4.1** Let $K$ be a compact self-adjoint operator in a Hilbert space $X$ with $\text{Ker}K = \{0\}$ and eigenvalues $\lambda_j(K), j = 1, 2, \ldots, \infty$. Assume that

$$\sum_{j=1}^{\infty} |\lambda_j(K)|' < \infty$$

then the completeness of the generalized eigenfunctions of $A + B$ holds.
for some \( r \geq 1 \), and let \( S \) be a compact operator in \( X \) such that \( I + S \) is invertible. Then the system of (generalized) eigenfunctions of the operator

\[
A := K(I + S)
\]

is complete in \( X \).

**THEOREM 4.1** Let \( A + B \) be defined in (2.12). Then the generalized eigenfunctions of \( A + B \) are complete in \( \mathcal{H} \).

**Proof.** Since \( A \) is a skew-adjoint operator with compact resolvents and 0, \( \infty \in \rho(A) \), \( (iA)^{-1} \) is compact, self-adjoint and \( \text{Ker}(iA)^{-1} = \{0\} \). Moreover, by (3.45), (3.46) (where we can take \( \gamma \equiv 0 \)), we have \( \{\lambda_k((iA)^{-1})\}_{k=1}^\infty \in l^2 \). Since

\[
(i(A + B))^{-1} = (iA)^{-1}(I + BA^{-1})^{-1} = (iA)^{-1}(I - BA^{-1}(I + BA^{-1})^{-1}),
\]

\( BA^{-1} \) and \( BA^{-1}(I + BA^{-1})^{-1} \) are compact and \( I - BA^{-1}(I + BA^{-1})^{-1} \) is invertible, the result then follows from Lemma 4.1. \( \square \)

Finally we show the Riesz basis property for the system (2.12), which is a more profound result for systems governed by partial differential equations. To do this, we need the following Theorem 4.2 of Wang et al. (2004).

**THEOREM 4.2** Let \( X \) be a separable Hilbert space, and \( A \) be the generator of a \( C_0 \)-semigroup \( T(t) \) on \( X \). Suppose the following conditions are fulfilled:

1. \( \sigma(A) = \sigma_1(A) \cup \sigma_2(A) \) and \( \sigma_2(A) = \{\lambda_k\}_{k=1}^\infty \) consists of isolated eigenvalues of finite algebraic multiplicity only;

2. \( \sup_{k \geq 1} m_a(\lambda_k) < \infty \), where \( m_a(\lambda_k) := \dim E(\lambda_k, A)x \) and \( E(\lambda_k, A) \) denotes the eigenprojection associated with \( \lambda_k \);

3. there is a constant \( \alpha \) such that \( \sup\{\Re \lambda | \lambda \in \sigma_1(A)\} \leq \alpha \leq \inf\{\Re \lambda | \lambda \in \sigma_2(A)\} \) and \( \inf_{n \neq m} |\lambda_n - \lambda_m| > 0 \).

Then the following assertions hold true:

(i) There exist two \( T(t) \)-invariant closed subspaces \( X_1 \) and \( X_2 \) such that \( \sigma(A|_{X_1}) = \sigma_1(A) \), \( \sigma(A|_{X_2}) = \sigma_2(A) \) and \( \{E(\lambda_k, A)|_X\}_{k=1}^\infty \) forms a Riesz basis for \( X_2 \), i.e.

\[
x_2 = \sum_{k=1}^\infty E(\lambda_k, A)x_2, \quad \forall x_2 \in X_2;
\]

(ii) there exist two positive constants \( C_1, C_2 \) independent of \( k \) and \( x_2 \) such that

\[
C_1 \sum_{k=1}^\infty \|E(\lambda_k, A)x_2\|^2 \leq \left\| \sum_{k=1}^\infty E(\lambda_k, A)x_2 \right\|^2 \leq C_2 \sum_{k=1}^\infty \|E(\lambda_k, A)x_2\|^2.
\]

Furthermore,

\[
X = X_1 \oplus X_2;
\]

(iii) if \( \sup_{k \geq 1} \|E(\lambda_k, A)\| < \infty \), then \( D(A) \subset X_1 \oplus X_2 \subset X \);
(iv) $X$ can decompose into the topological direct sum $X = X_1 \oplus X_2$ if and only if
\[ \sup_{n \geq 1} \left\| \sum_{k=1}^{n} E(\lambda_k, A) \right\| < \infty. \]

Combining Theorem 4.1, 4.2 and 3.4, we get the following Theorem 4.3.

**THEOREM 4.3** System (2.12) is a Riesz spectral system (in the sense that its generalized eigenfunctions form a Riesz basis for $\mathcal{H}$, see Curtain & Zwart, 1995). Moreover, the spectrum-determined growth condition holds true, i.e. $s(A + B) = \omega(A + B)$, where $s(A + B) := \sup\{\text{Re}\lambda|\lambda \in \sigma(A + B)\}$ is the spectral bound of $A + B$ and $\omega(A + B)$ is the growth bound of the semigroup $e^{(A+B)t}$ generated by $A + B$.

**Proof.** In view of Theorem 3.4, we may take $\sigma_2(A + B) := \sigma(A + B)$ and $\sigma_1(A + B) := \{-\infty\}$. Then conditions 1–3 of Theorem 4.2 with $A := A + B$, $X := \mathcal{H}$ are satisfied. Moreover, Theorem 4.1 implies that $X_1 = \{0\}$. Thus, the first assertion of Theorem 4.2 says that there is a sequence of the generalized eigenfunctions of $A + B$, which forms a Riesz basis for $\mathcal{H}$. Finally, the spectrum-determined growth condition can be deduced directly from the Riesz basis generation and the algebraic simplicity of the high eigenvalues. □

Finally, we discuss the stability of the system (2.12). First from (3.49), (3.50), in order to achieve the exponential stability, it is necessary to require that
\[ \int_0^1 \gamma(\xi) d\xi > 0. \]  \hspace{1cm} (4.1)

Condition (4.1) guarantees that in (3.47), (3.48):
\[ -\frac{1}{4} \frac{\delta(r_2^2 - r_1^2) + \sqrt{\delta^2(r_1^2 - r_2^2)^2 + (\delta^2 - \delta)r_1^2r_2^2}}{\sqrt{\delta^2(r_1^2 - r_2^2)^2 + (\delta^2 - \delta)r_1^2r_2^2}} \int_0^1 \gamma(\xi) d\xi < 0 \]

and
\[ -\frac{1}{4} \frac{\delta(r_2^2 - r_1^2) + \sqrt{\delta^2(r_1^2 - r_2^2)^2 + (\delta^2 - \delta)r_1^2r_2^2}}{\sqrt{\delta^2(r_1^2 - r_2^2)^2 + (\delta^2 - \delta)r_1^2r_2^2}} \int_0^1 \gamma(\xi) d\xi < 0. \]

These in turn imply that the high eigenvalues of the system (2.12) are located on the left half complex plane.

Next, suppose $\gamma \geq 0$ and $\gamma(x)|_I > 0$ for $x \in I$, a measurable subset of $[0, 1]$ with positive measure. Then for each $Y := [z, w, u, v] \in D(A + B)$, it has
\[ \langle (A + B)Y, Y \rangle_{\mathcal{H}} = \int_0^1 \left[ (a_1z'' + a_2u'')w + (a_2z'' + a_3u'')v ight. \\
\left. - \gamma(\xi)|w|^2 + \left\langle \gamma' \begin{pmatrix} w' \\ u' \end{pmatrix}, \begin{pmatrix} z' \\ u' \end{pmatrix} \right\rangle_{\mathcal{C}^2} \right] d\xi \]
\[= a_1 z' \overline{w}_0 + a_2 u' \overline{w}_0 + a_3 u' \overline{v}_0 - \int_0^1 \gamma(\xi)|w|^2 \, d\xi + a_2 z' \overline{w}_0 \]

\[+ \int_0^1 \left[ \left< \Upsilon \left( \begin{array}{c} w' \\ u' \end{array} \right), \left( \begin{array}{c} z' \\ u' \end{array} \right) \right>_{C^2} - \left< \Upsilon \left( \begin{array}{c} z' \\ v' \end{array} \right), \left( \begin{array}{c} w' \\ v' \end{array} \right) \right>_{C^2} \right] \, d\xi \]

and thus

\[\text{Re}(\langle A + B \rangle Y, Y)_{\mathcal{H}} = - \int_0^1 \gamma(\xi)|w|^2 \, d\xi \leq 0.\]

So the operator \(A + B\) defined in (2.12) is dissipative and hence the real parts of all the eigenvalues are non-positive, i.e. \(\text{Re}(\lambda(A + B)) \leq 0\), and \(e^{(A+B)t}\) is a \(C_0\)-semigroup of contractions.

Thirdly, let \(\lambda := i\tau, 0 \neq \tau \in \mathbb{R}\) be an eigenvalue of \(A + B\) and let \(Y\) be its corresponding eigenfunction. From previous arguments, one has

\[0 \equiv \text{Re}(\langle (A + B) Y, Y \rangle_{\mathcal{H}}) = - \int_0^1 \gamma(\xi)|w|^2 \, d\xi \]

and hence \(w \equiv 0\) in \(I\) and so is \(w \equiv 0\) in \([0, 1]\) (see Guo, 2002). By \((A + B)Y = i\tau Y\), we further have that \(z \equiv 0\) and

\[
\begin{align*}
  u''(x) &= 0, & 0 < x < 1, \\
  \rho_\tau \tau^2 u(x) + a_3 u''(x) &= 0, & 0 < x < 1, \\
  u(0) &= u'(1) = 0.
\end{align*}
\]

A direct computation yields \(v = u \equiv 0\). Thus \(Y \equiv 0\). This contradicts the fact that \(\lambda = i\tau\) is an eigenvalue of \(A + B\). Therefore, there is no eigenvalue on the imaginary axis and hence the system (2.12) is exponentially stable.

**Theorem 4.4** Let \(A + B\) be defined by in (2.12). Suppose \(\gamma \geq 0\) and \(\gamma(x)|_I > 0\) for \(x \in I\), a measurable subset of \([0, 1]\) with positive measure. Then the system (2.12) is exponentially stable.

To end this section, we point out an interesting case where condition (4.1) is satisfied but continuous function \(\gamma\) may change sign in \([0, 1]\) (see Wang et al., 2005). Let

\[\gamma_+(x) := \max\{\gamma(x), 0\}, \quad \gamma_-(x) := \max\{\gamma(x), 0\}\]

and define bounded operators \(B_\pm\) by

\[
B_\pm \begin{bmatrix} z \\ w \\ u \\ v \end{bmatrix}^T := \begin{bmatrix} 0 & -\gamma_\pm w \\ -\gamma_\pm w & 0 \\ 0 & 0 \end{bmatrix}^T, \quad \forall \begin{bmatrix} z \\ w \\ u \\ v \end{bmatrix} \in \mathcal{H}.
\]

Then \(A + B\) can be written into \(A + B = A + B_+ + B_-\).

**Theorem 4.5** Suppose condition (4.1) and

\[
\max_{x \in [0, 1]} \{-\gamma_-(x)\} < |s(A + B_+)|.
\]

Then the system (2.12) is exponentially stable.
Proof. It is easy to verify that $B_-$ is a self-adjoint operator and
\[ \|B_-\| = \max_{x \in [0, 1]} \{-\gamma_-(x)\}. \]

By Theorem 4.4, $e^{(A + B_+)^t}$ is a $C_0$-semigroup of contractions and $s(A + B_+) < 0$. By perturbation theory of semigroup of linear operators (see Theorem 1.1 of Pazy, 1983, pp. 76), we have $\lambda \in \rho(A + B)$ whenever $\text{Re}\lambda > s(A + B_+) + \|B_-\|$. Again, Theorem 4.3 guarantees that
\[ \omega(A + B) = s(A + B) \leq s(A + B_+) + \|B_-\| < 0. \]

Therefore, the system (2.12) is exponentially stable. \qed

5. Concluding remarks

In this article, we impose only one internal viscous damping for a 1D system of mixed swelling porous elastic soils and fluid and show that the whole system can be exponentially stabilized by designed damping. This is a first attempt to stabilize a system of two coupled wave equations by only one distributed control. Our result is much beyond the stabilization itself. In other words, we show that the solution of the system can be expanded in terms of vibration frequencies and the eigenfrequencies are sufficient to determine the stability of the whole system. The effect of the variable feedback proportional coefficients is clearly demonstrated by the asymptotical behaviours of the eigenpairs. The results have potential applications to the systems of swelling of soils, plants, drying of fibres, wood, paper, etc. That is, it is enough to change the viscoelasticity of one material by smart materials for the purpose of the vibration suppression of the whole coupled system. Moreover, the techniques used in the article can be used for the analysis of dynamic and control for such a kind of systems.

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