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Stability of a nonuniform Rayleigh beam with indefinite damping

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Abstract

This is a continuation of our earlier work [J.M. Wang, G.Q. Xu, S.P. Yung, Exponential stability for variable coefficients Rayleigh beams under boundary feedback control: a Riesz basis approach, Systems Control Lett. 51 (1) (2004) 33–50] on the study of a nonhomogeneous Rayleigh beam and this time the stabilization is achieved via an internal damping instead of the boundary feedbacks. We continue to address a conjecture of Guo [Basis property of a Rayleigh beam with boundary stabilization, J. Optim. Theory Appl. 112(3) (2002) 529–547] in this paper and demonstrate how the damping term can affect the decay rate asymptotically. By a detailed spectral analysis, we obtain a necessary condition for the stability and establish the Riesz basis property as well as the spectrum determined growth condition for the system. Furthermore, when the damping is indefinite, we provide a condition on how "negative" the damping can be without destroying the exponential stability.
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1. Introduction

In this paper, we study the following variable coefficient Rayleigh beam with an indefinite damping term:

$$\begin{cases} \rho(x)\dfrac{\partial^2 u}{\partial t^2} - \dfrac{\partial}{\partial x}\left(I_\rho(x)\dfrac{\partial^3 u}{\partial t^2 \partial x}\right) + \dfrac{\partial^2}{\partial x^2}\left(EI(x)\dfrac{\partial^2 u}{\partial x^2}\right) \\ \quad -\dfrac{\partial}{\partial x}\left(a(x)\dfrac{\partial^2 u}{\partial x \partial t}\right) = 0, \quad 0 < x < 1, \quad t > 0, \\ u(0,t) = u(1,t) = \dfrac{\partial u}{\partial x}(0,t) = \dfrac{\partial u}{\partial x}(1,t) = 0, \quad t > 0, \\ u(x,0) = u_0(x), \quad \dfrac{\partial u}{\partial t}(x,0) = u_1(x). \end{cases} \tag{1.1}$$

Here, $u(x,t)$ is the transverse displacement, and x and t stand for the spatial position and time, respectively. The coefficient functions $\rho(x) > 0$ is the mass density, $EI(x) > 0$ is the stiffness of the beam, and $I_\rho(x) > 0$ is the mass moment of inertia. Damping $a(x)$ is a continuously differentiable function. The same problem was investigated in [21] except that there was no damping term but two boundary feedbacks. We refer the readers to [16,21] for further details of this model. The uniform and nonuniform Rayleigh problem with boundary feedback controls were discussed, respectively, in [5,21], where a conjecture on how the control parameter affects the decay rate was proposed and then answered, respectively. We continue to investigate this conjecture here for the damping cases and will reveal the relationship between the internal damping term and the asymptotic decay rate. As a consequence, we obtain the following condition:

$$\int_0^1 \frac{a(x)}{EI(x)}\left(\frac{I_\rho(x)}{EI(x)}\right)^{-1/2} \mathrm{d}x > 0, \tag{1.2}$$

as a necessary condition to make the system exponentially stable. This condition will allow $a(x)$ to be indefinite in the interval [0, 1] and indicate how indefinite the damping could be without destroying the exponential stability. Same can be said for $I_\rho(x)$ and $EI(x)$. For convenience, we always assume that

$$\rho(x), I_\rho(x), EI(x) \in C^4[0,1]. \tag{1.3}$$

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There are many references on the stability of systems with indefinite dampings. For instance, articles [2,3,10] dealt with wave equations and [14] studied the constant-coefficient version of (1.1). In there, all coefficients were assumed to be constants except for $a(x)$. It was shown that the system is exponentially stable under the condition that there exists α and β so that $0 \leqslant \alpha < a(x) < \beta < \infty$ or $\varepsilon a(x)$ with $\varepsilon \to 0$. Under the perspective of operator theory, systems with indefinite dampings can be regarded as a bounded perturbation from those of the free systems. There are two commonly used approaches to study these perturbed systems. The first one is due to Huang [9], which says that if the resolvent $R(\lambda, A+B)$ is uniformly bounded along the imaginary axis, then $A+B$ generates an exponentially stable semigroup on the energy space (see [10]). The second one is due to [15,24], which say that the semigroup generated by $A+B$ with bounded linear operator B will satisfy the spectrum determined growth condition if the generator A (not necessarily skew-adjoint) satisfies the following property:

There exists $N>0$ such that the spectrum $\sigma(A)=\{\lambda_n\}$ of A is separated and simple when $|\lambda_n| \geqslant N$ and there is a sequence of the generalized eigenvectors of A that forms a Riesz basis in the state Hilbert space $\mathscr{H}$.

There is a third approach, namely the Riesz basis approach, which shows that the generalized eigenfunctions of the system form a Riesz basis, and then deduces the spectrum determined growth condition and various stability results from the eigenvalue distribution of the system (see [5–7,21–23]). This approach is a bit complicated than the former two, but will be the approach that we pursue in this paper because the other two do not yield any spectral information in addressing our perspectives. Although the characteristic equation this time is not in the classical form of [1,12], and so the techniques of Birkhoff [1] (and later [12]) that have been so successful on the applications of Euler–Bernoulli beams (see [6,22]) cannot be applied here, this obstacle can be overcome by invoking the ideas of operator pencil from [17–19]. Eventually, asymptotic expressions of the eigenvalues of system (1.1) are obtained and the Riesz basis property of the generalized eigenfunctions of the system is deduced. These will imply the spectrum determined growth condition and eventually stability is established.

The rest of this paper is organized as follows. In Section 2, we will find a suitable Hilbert space framework for system (1.1), and then the system will be shown to generate a C_0-group. We then study its eigenvalue problem. The main trick is to use a space-scaling transformation to derive an equivalent eigenvalue boundary problem and this leads to much simpler asymptotic expansions. In Section 2, we shall apply the techniques in [17–19] to the fundamental solutions of the eigenvalue boundary problem, and then use the results to expand the characteristic determinant in deducing the asymptotic behavior of the eigenvalues. In the last section, we shall reveal how the damping term affects the decay rate asymptotically, and then discuss how negative the damping could be so that the system is still exponentially stable. Quite a lot of steps in our investigation coincide with those in [21]. So we shall call on them whenever necessary.

2. State space and eigenvalue problem setup

We start our investigation by formulating the problem in an appropriate Hilbert space. Let $H_0^1(0,1)$ and $H_0^2(0,1)$ be the usual two Sobolev spaces and let $\mathscr{P}: H_0^1(0,1) \to H^{-1}(0,1)$ be a linear operator defined by (see [8]):

$$\mathscr{P}u := \rho(x)u(x) - (I_\rho(x)u'(x))', \quad \forall u \in H_0^1(0,1).$$

Then $\mathscr{P}^{-1}$ exists and is an isomorphism from $H^{-1}(0,1)$ to $H_0^1(0,1)$ (see [8]). We denote by $\mathscr{H}$ the complex Hilbert space

$$\mathscr{H} := H_0^2(0,1) \times H_0^1(0,1) \tag{2.1}$$

and endow with the following inner product, for any $[f,g],[u,v] \in \mathscr{H}$,

$$\langle [f,g],[u,v]\rangle_{\mathscr{H}} := \int_0^1 [EI(x)f''(x)\overline{u''(x)} + I_\rho(x)g'(x)\overline{v'(x)} + \rho(x)g(x)\overline{v(x)}]\,\mathrm{d}x. \tag{2.2}$$

In $\mathscr{H}$, define two linear operators $\mathscr{A}$ and $\mathscr{B}$, respectively, by

$$\begin{cases} \mathscr{A}[f,g] := [g, -\mathscr{P}^{-1}((EI(x)f''(x))'')] \in \mathscr{H}, \\ \quad \forall [f,g] \in \mathscr{D}(\mathscr{A}), \\ \mathscr{D}(\mathscr{A}) := \{[f,g] \in \mathscr{H} \mid f \in H^3(0,1), \quad g \in H_0^2(0,1)\} \end{cases} \tag{2.3}$$

and

$$\mathscr{B}[f,g] := [0, \ \mathscr{P}^{-1}((a(x)g'(x))')], \quad \forall [f,g] \in \mathscr{D}(\mathscr{B}) = \mathscr{H}. \tag{2.4}$$

If we let $v := u_t$, $U(t) := [u(t), v(t)]$, then system (1.1) can be formulated into an abstract evolution equation in $\mathscr{H}$:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} U(t) = (\mathscr{A}+\mathscr{B})U(t), \quad t>0, \\ U(0) = [u_0, u_1]. \end{cases} \tag{2.5}$$

Lemma 2.1. *In $\mathscr{H}$, $\mathscr{A}$ is a skew-adjoint operator with compact resolvents and $\mathscr{B}$ is a bounded operator. Hence, $\mathscr{A}+\mathscr{B}$ generates a C_0-group $\mathrm{e}^{(\mathscr{A}+\mathscr{B})t}$ and the spectrum $\sigma(\mathscr{A}+\mathscr{B})$ consists of isolated eigenvalues only.*

Proof. It is easy to see that $\mathscr{A}$ is a skew-adjoint operator with compact resolvents and $\mathscr{B}$ is bounded. A standard perturbation result (see [13]) will then imply that $\mathscr{A}+\mathscr{B}$ has compact resolvents and generates a C_0-group $\mathrm{e}^{(\mathscr{A}+\mathscr{B})t}$ on $\mathscr{H}$. Thus, the spectrum $\sigma(\mathscr{A})$ has isolated eigenvalues only. □

To calculate the eigenvalue of $\mathscr{A}+\mathscr{B}$, we convert the eigenvalue problem into a more convenient form. Let $\lambda \in \sigma(\mathscr{A}+\mathscr{B})$, and let $[\phi,\psi] \in \mathscr{H}$ be such that

$$(\mathscr{A}+\mathscr{B})[\phi,\psi] = \lambda[\phi,\psi].$$

Then, we have $\psi = \lambda\phi$ with ϕ satisfying

$$\begin{cases} \lambda^2\rho(x)\phi(x) - \lambda^2(I_\rho(x)\phi'(x))' + (EI(x)\phi''(x))'' \\ \quad -\lambda(a(x)\phi'(x))' = 0, \quad 0 < x < 1, \\ \phi(0) = \phi(1) = \phi'(0) = \phi'(1) = 0. \end{cases} \tag{2.6}$$

We need the following lemma in our later discussion.

Lemma 2.2. *Let $h_1(x), h_2(x)$ be two linearly independent solutions for the second order linear homogeneous differential equation*

$$(I_\rho(x)\phi'(x))' - \rho(x)\phi(x) = 0. \tag{2.7}$$

Then

$$D := h_1(0)h_2(1) - h_1(1)h_2(0) \neq 0. \tag{2.8}$$

Proof. Assume not, then the following system of linear equations:

$$\begin{cases} t_1h_1(0) + t_2h_2(0) = 0, \\ t_1h_1(1) + t_2h_2(1) = 0 \end{cases}$$

is singular because the determinant of the coefficient matrix is $h_1(0)h_2(1) - h_1(1)h_2(0) = 0$. So there exists a nontrivial solution, say $[c_1,\ c_2]$. Let $z := c_1h_1 + c_2h_2$, then z is a solution of the differential equation with two boundary conditions:

$$\begin{cases} (I_\rho(x)z'(x))' - \rho(x)z(x) = 0, \\ z(0) = z(1) = 0. \end{cases} \tag{2.9}$$

Introduce a second differential operator $\mathscr{T}$ in $L^2(0,1)$ by

$$\begin{cases} \mathscr{T}z := \dfrac{(I_\rho(x)z'(x))'}{\rho(x)}, \quad \forall z \in \mathscr{D}(\mathscr{T}), \\ \mathscr{D}(\mathscr{T}) = \{z \in H^2(0,1) \mid z(0) = z(1) = 0\} \end{cases} \tag{2.10}$$

under a compatible inner product in $L^2(0,1)$ by

$$\langle f, g\rangle := \int_0^1 \rho(x)f(x)\overline{g(x)}\,dx, \quad \forall f, g \in L^2(0,1).$$

Then $\mathscr{T}$ is densely defined, closed, and negative definite in $L^2(0,1)$. Thus $(I - \mathscr{T})^{-1}$ exists and bounded in $L^2(0,1)$. Therefore, (2.9) can only have a trivial solution, i.e., $z \equiv 0$. So h_1 and h_2 are linearly dependent and this contradicts the assumption of the lemma. The proof is then complete. □

To further simplify (2.6), we expand it into the following form:

$$\begin{cases} \phi^{(4)}(x) + 2\dfrac{EI'(x)}{EI(x)}\phi'''(x) + \dfrac{EI''(x)}{EI(x)}\phi''(x) \\ \quad -\lambda^2\left(\dfrac{I_\rho(x)}{EI(x)}\phi''(x) + \dfrac{I'_\rho(x)}{EI(x)}\phi'(x) - \dfrac{\rho(x)}{EI(x)}\phi(x)\right) \\ \quad -\lambda\left(\dfrac{a(x)}{EI(x)}\phi''(x) + \dfrac{a'(x)}{EI(x)}\phi'(x)\right) = 0, \\ \phi(0) = \phi(1) = \phi'(0) = \phi'(1) = 0. \end{cases} \tag{2.11}$$

If we introduce a space-scaling transformation (see [6,21,22])

$$\phi(x) := f(z), \quad z := \frac{1}{h}\int_0^x \left(\frac{I_\rho(\zeta)}{EI(\zeta)}\right)^{1/2} d\zeta,$$
$$h := \int_0^1 \left(\frac{I_\rho(\zeta)}{EI(\zeta)}\right)^{1/2} d\zeta, \tag{2.12}$$

then (2.11) can be rewritten as

$$\begin{cases} f^{(4)}(z) + a_1(z)f'''(z) + a_2(z)f''(z) + a_3(z)f'(z) \\ \quad -h^2\lambda^2[f''(z) + b_1(z)f'(z) - b_2(z)f(z)] \\ \quad -\lambda[c_0(z)f''(z) + c_1(z)f'(z)] = 0, \\ f(0) = f(1) = f'(0) = f'(1) = 0. \end{cases} \tag{2.13}$$

Here, $f' := df/dz$, z_x denotes the derivative dz/dx, and

$$a_1(z) := 6\frac{z_{xx}}{z_x^2} + 2\frac{1}{z_x}\frac{EI'(x)}{EI(x)}, \quad z_x = \frac{1}{h}\left(\frac{I_\rho(x)}{EI(x)}\right)^{1/2},$$
$$a_2(z) := 3\frac{z_{xx}^2}{z_x^4} + 4\frac{z_{xxx}}{z_x^3} + 6\frac{z_{xx}}{z_x^3}\frac{EI'(x)}{EI(x)} + \frac{1}{z_x^2}\frac{EI''(x)}{EI(x)},$$
$$a_3(z) := \frac{z_{xxxx}}{z_x^4} + 2\frac{z_{xxx}}{z_x^4}\frac{EI'(x)}{EI(x)} + \frac{z_{xx}}{z_x^4}\frac{EI''(x)}{EI(x)},$$
$$b_1(z) := \frac{z_{xx}}{z_x^2} + \frac{1}{h^2z_x^3}\frac{I'_\rho(x)}{EI(x)}, \quad b_2(z) := \frac{1}{h^2z_x^4}\frac{\rho(x)}{EI(x)},$$
$$c_0(z) := \frac{1}{z_x^2}\frac{a(x)}{EI(x)}, \quad c_1(z) := \frac{z_{xx}}{z_x^4}\frac{a(x)}{EI(x)} + \frac{1}{z_x^3}\frac{a'(x)}{EI(x)}. \tag{2.14}$$

Now if we replace λ by $\mu := h\lambda$, then (2.13) is changed to

$$\begin{cases} f^{(4)}(z) + a_1(z)f'''(z) + a_2(z)f''(z) + a_3(z)f'(z) \\ \quad -\mu^2[f''(z) + b_1(z)f'(z) - b_2(z)f(z)] \\ \quad -\mu\left[\dfrac{1}{h}c_0(z)f''(z) + \dfrac{1}{h}c_1(z)f'(z)\right] = 0, \\ f(0) = f(1) = f'(0) = f'(1) = 0. \end{cases} \tag{2.15}$$

All these can be summarized into the following theorem.

Theorem 2.1. $\lambda \in \sigma(\mathscr{A} + \mathscr{B})$ *if and only if* (2.15) *has a trivial solution $f(z)$ for $\mu := h\lambda$. In addition, the function $\phi(x)$ in the eigenfunction pair $[\phi, \lambda\phi]$ of $\mathscr{A} + \mathscr{B}$ is given by* (2.12).

3. Asymptotic expressions of eigen-frequencies

The characteristic equation (2.15) is not in the classical form in the sense that the highest power of the eigenvalue parameter μ is not equal to the order of the highest derivative of the equation. This makes the results of [1,12] not applicable. Fortunately, the results of [17–19] can be applied and this is how we will proceed. Due to Lemma 2.1 and the fact that the eigenvalues of (2.15) (or (2.13)) are symmetric about the real axis, we need only consider the eigenvalues located in the upper half plane:

$$\mathscr{S} := \{z \in \mathbb{C} : 0 \leqslant \arg z \leqslant \pi\}. \tag{3.1}$$

Let $\mu = \mathrm{i}\rho$. Then it is sufficient to consider $\rho \in \mathscr{S}_0$ with

$$\mathscr{S}_0 := \left\{ z \in \mathbb{C} : -\frac{\pi}{2} \leqslant \arg z \leqslant \frac{\pi}{2} \right\}. \tag{3.2}$$

Let

$$\omega_1 := \mathrm{i}, \quad \omega_2 := -\mathrm{i}, \tag{3.3}$$

where both ω_1 and ω_2 are the square roots of -1. To pursue the asymptotic estimates for the eigenvalues, we need the following lemma.

Lemma 3.1. *For $\rho \in \mathscr{S}_0$, with $|\rho|$ sufficiently large, the equation*

$$\begin{aligned} &f^{(4)}(z) + a_1(z) f'''(z) + a_2(z) f''(z) + a_3(z) f'(z) \\ &\quad + \rho^2 [f''(z) + b_1(z) f'(z) - b_2(z) f(z)] \\ &\quad - \mathrm{i}\rho \left[\frac{1}{h} c_0(z) f''(z) + \frac{1}{h} c_1(z) f'(z) \right] = 0 \end{aligned} \tag{3.4}$$

has four linearly independent fundamental solutions $y_s(z;\rho)$ $(s = 1, 2, 3, 4)$ and they possess the following asymptotic expressions (for $j = 0, 1, 2, 3$)

$$y_s^{(j)}(z;\rho) = h_s^{(j)}(z) + \mathcal{O}(\rho^{-1}), \quad s = 1, 2, \tag{3.5}$$

$$y_s^{(j)}(z;\rho) = (\rho\omega_{s-2})^j \, \mathrm{e}^{\rho\omega_{s-2} x} [y_s(z) + \mathcal{O}(\rho^{-1})], \quad s = 3, 4. \tag{3.6}$$

Here, for $s = 3, 4$,

$$y_s(z) := \exp\left(\int_0^z \frac{1}{2} \left[b_1(t) - a_1(t) - \frac{\mathrm{i}}{h} \omega_{s-2} c_0(t) \right] \mathrm{d}t \right), \tag{3.7}$$

$$y_s(0) = 1, \quad y_s(1) = \exp(D_1 + \mathrm{i}\omega_{s-2} D_2), \tag{3.8}$$

$$D_1 := \frac{1}{2} \int_0^1 [b_1(t) - a_1(t)] \, \mathrm{d}t, \quad D_2 := -\frac{1}{2h} \int_0^1 c_0(t) \, \mathrm{d}t \tag{3.9}$$

and $h_1(z) := h_1(x(z)), h_2(z) := h_2(x(z))$ are two linearly independent solutions to the following equation:

$$f''(z) + b_1(z) f'(z) - b_2(z) f(z) = 0.$$

Proof. This is a direct consequence of Tretter's results of [19] and a detailed proof can be found in [20]. □

We shall use the following notation in the sequel:

$$[a]_1 := a + \mathcal{O}(\rho^{-1}).$$

Using the asymptotic expansions of the fundamental solutions in (3.5)–(3.9), asymptotic expansions of the boundary conditions (2.15) can be obtained.

Lemma 3.2. *Denote the boundary conditions of system* (2.15) *by U_1, U_2, U_3 and U_4, then for $\rho \in \mathscr{S}_0$, with $|\rho|$ sufficiently large, we have the following asymptotic expansions,*

$$\begin{aligned} U_4(y_s;\rho) &= y_s(0;\rho) \\ &= \begin{cases} h_s(0) + \mathcal{O}(\rho^{-1}) := h_s(0)[1]_1, & s = 1, 2, \\ 1 + \mathcal{O}(\rho^{-1}) := [1]_1, & s = 3, 4, \end{cases} \end{aligned} \tag{3.10}$$

$$\begin{aligned} U_3(y_s;\rho) &= y_s'(0;\rho) \\ &= \begin{cases} x_z(0) h_s'(0) + \mathcal{O}(\rho^{-1}) \\ \quad := x_z(0) h_s'(0)[1]_1, & s = 1, 2, \\ \rho\omega_{s-2}(1 + \mathcal{O}(\rho^{-1})) \\ \quad := \rho\omega_{s-2}[1], & s = 3, 4, \end{cases} \end{aligned} \tag{3.11}$$

$$\begin{aligned} U_2(y_s;\rho) &= y_s(1;\rho) \\ &= \begin{cases} h_s(1) + \mathcal{O}(\rho^{-1}), & s = 1, 2, \\ \mathrm{e}^{\rho\omega_{s-2}} (\exp(D_1 + \mathrm{i}\omega_{s-2} D_2) \\ \quad + \mathcal{O}(\rho^{-1})), & s = 3, 4, \end{cases} \\ &:= \begin{cases} h_s(1)[1]_1, & s = 1, 2, \\ \mathrm{e}^{\rho\omega_{s-2}} \exp(D_1 + \mathrm{i}\omega_{s-2} D_2)[1]_1, & s = 3, 4, \end{cases} \end{aligned} \tag{3.12}$$

$$\begin{aligned} U_1(y_s;\rho) &= y_s'(1;\rho) \\ &= \begin{cases} x_z(1) h_s'(1) + \mathcal{O}(\rho^{-1}), & s = 1, 2, \\ \rho\omega_{s-]2} \mathrm{e}^{\rho\omega_{s-2}} (\exp(D_1 + \mathrm{i}\omega_{s-2} D_2) \\ \quad + \mathcal{O}(\rho^{-1})), & s = 3, 4, \end{cases} \\ &:= \begin{cases} x_z(1) h_s'(1)[1]_1, & s = 1, 2, \\ \rho\omega_{s-2} \mathrm{e}^{\rho\omega_{s-2}} \exp(D_1 + \mathrm{i}\omega_{s-2} D_2)[1]_1, & s = 3, 4. \end{cases} \end{aligned} \tag{3.13}$$

Proof. The proof is a direct substitution of (3.5)–(3.9) into the boundary conditions of (2.15). □

If we substitute (3.10)–(3.13) into the characteristic determinant of (2.15) with $\mu = \mathrm{i}\rho$ for any $\rho \in \mathscr{S}_0$, then

$$\Delta(\rho) := \begin{vmatrix} U_4(y_1,\rho) & U_4(y_2,\rho) & U_4(y_3,\rho) & U_4(y_4,\rho) \\ U_3(y_1,\rho) & U_3(y_2,\rho) & U_3(y_3,\rho) & U_3(y_4,\rho) \\ U_2(y_1,\rho) & U_2(y_2,\rho) & U_2(y_3,\rho) & U_2(y_4,\rho) \\ U_1(y_1,\rho) & U_1(y_2,\rho) & U_1(y_3,\rho) & U_1(y_4,\rho) \end{vmatrix} \tag{3.14}$$

and we arrive at the following theorem.

Theorem 3.1. *In sector $\mathscr{S}_0$, the characteristic determinant $\Delta(\rho)$ of the characteristic equation* (2.15) *has an asymptotic expansion*

$$\Delta(\rho) = \rho^2 D \{ \mathrm{e}^{\rho\omega_1} \mathrm{e}^{D_1 - D_2} - \mathrm{e}^{\rho\omega_2} \mathrm{e}^{D_1 + D_2} + \mathcal{O}(\rho^{-1}) \}, \tag{3.15}$$

where $D := (h_1(0) h_2(1) - h_1(1) h_2(0))$ as defined in (2.8), *$D_1 := \frac{1}{2} \int_0^1 (b_1(t) - a_1(t)) \, \mathrm{d}t$ and $D_2 := -\frac{1}{2h} \int_0^1 c_0(t) \, \mathrm{d}t$ are defined in* (3.9). *As a by-product, the boundary problem* (2.15) *is strongly regular in the sense* of [18, Definition 2.7]. *Therefore, the zeros of $\Delta(\rho)$ are simple when their moduli are sufficiently large.*

Proof. Substituting (3.10)–(3.13) into the characteristic determinant (3.14), we have

$$\Delta(\rho)=\begin{vmatrix} h_1(0)[1]_1 & h_2(0)[1]_1 & [1]_1 & [1]_1 \\ x_z(0)h_1'(0)[1]_1 & x_z(0)h_2'(0)[1]_1 & \rho\omega_1[1] & \rho\omega_2[1] \\ h_1(1)[1]_1 & h_2(1)[1]_1 & e^{\rho\omega_1}e^{D_1+i\omega_1 D_2}[1]_1 & e^{\rho\omega_2}e^{D_1+i\omega_2 D_2}[1]_1 \\ x_z(1)h_1'(1)[1]_1 & x_z(1)h_2'(1)[1]_1 & \rho\omega_1 e^{\rho\omega_1}e^{D_1+i\omega_1 D_2}[1]_1 & \rho\omega_2 e^{\rho\omega_2}e^{D_1+i\omega_2 D_2}[1]_1 \end{vmatrix}.$$

Since $\omega_1=i$ and $\omega_2=-i$ in sector $\mathscr{S}_0$, we have

$$\begin{aligned}\Delta(\rho)&=\begin{vmatrix} h_1(0)[1]_1 & h_2(0)[1]_1 & [1]_1 & [1]_1 \\ x_z(0)h_1'(0)[1]_1 & x_z(0)h_2'(0)[1]_1 & i\rho[1] & -i\rho[1] \\ h_1(1)[1]_1 & h_2(1)[1]_1 & e^{\rho\omega_1}e^{D_1-D_2}[1]_1 & e^{\rho\omega_2}e^{D_1+D_2}[1]_1 \\ x_z(1)h_1'(1)[1]_1 & x_z(1)h_2'(1)[1]_1 & i\rho e^{\rho\omega_1}e^{D_1-D_2}[1]_1 & -i\rho e^{\rho\omega_2}e^{D_1+D_2}[1]_1 \end{vmatrix}\\
&=-i\rho\begin{vmatrix} h_1(0)[1]_1 & h_2(0)[1]_1 & [1]_1 \\ h_1(1)[1]_1 & h_2(1)[1]_1 & e^{\rho\omega_2}e^{D_1+D_2}[1]_1 \\ x_z(1)h_1'(1)[1]_1 & x_z(1)h_2'(1)[1]_1 & -i\rho e^{\rho\omega_2}e^{D_1+D_2}[1]_1 \end{vmatrix}\\
&\quad-i\rho\begin{vmatrix} h_1(0)[1]_1 & h_2(0)[1]_1 & [1]_1 \\ h_1(1)[1]_1 & h_2(1)[1]_1 & e^{\rho\omega_1}e^{D_1-D_2}[1]_1 \\ x_z(1)h_1'(1)[1]_1 & x_z(1)h_2'(1)[1]_1 & i\rho e^{\rho\omega_1}e^{D_1-D_2}[1]_1 \end{vmatrix}+\mathcal{O}(\rho)\\
&=\rho^2\{(e^{\rho\omega_1}e^{D_1-D_2}-e^{\rho\omega_2}e^{D_1+D_2})(h_1(0)h_2(1)-h_1(1)h_2(0))+\mathcal{O}(\rho^{-1})\}\\
&=\rho^2 D\{e^{\rho\omega_1}e^{D_1-D_2}-e^{\rho\omega_2}e^{D_1+D_2}+\mathcal{O}(\rho^{-1})\},\end{aligned}$$

which is (3.15). The strong regularity defined in [18, Definition 2.7] together with the simplicity of the zeros, when their moduli are large enough can be verified directly from the fact that $e^{D_1-D_2}>0$, $e^{D_1+D_2}>0$ and (2.8). The proof is then complete. □

Theorem 3.2. *Let* $\sigma(\mathscr{A}):=\{\lambda_k,\overline{\lambda}_k,\ k\in\mathbb{N}\}$. *Then for large enough* k, *the eigenvalues* λ_k *of problem* (2.6) *are simple and possess the following asymptotic expressions*

$$\lambda_k=\frac{1}{h}\left(-\frac{1}{2h}\int_0^1 c_0(t)\,dt+k\pi i\right)+\mathcal{O}(k^{-1}),\quad k\geqslant N, \tag{3.16}$$

where N *is a sufficiently large positive integer,* $h:=\int_0^1(I_\rho(\zeta)/EI(\zeta))^{1/2}\,d\zeta$ *is defined in* (2.12) *and* $c_0(z):=(1/z_x^2)(a(x)/EI(x))$ *is defined in* (2.14). *Thus,*

$$\int_0^1 c_0(t)\,dt=\int_0^1\frac{1}{z_x^2}\frac{a(x)}{EI(x)}\,dt=h\int_0^1\frac{a(x)}{EI(x)}\left(\frac{I_\rho(x)}{EI(x)}\right)^{-1/2}dx \tag{3.17}$$

and

$$\operatorname{Re}\lambda_k\to-\frac{1}{2h}\int_0^1\frac{a(x)}{EI(x)}\left(\frac{I_\rho(x)}{EI(x)}\right)^{-1/2}dx\quad\textit{for } k\to\infty. \tag{3.18}$$

Proof. In sector $\mathscr{S}_0$, we conclude from (3.15) and $\Delta(\rho)=0$ that

$$e^{\rho\omega_1}e^{D_1-D_2}-e^{\rho\omega_2}e^{D_1+D_2}+\mathcal{O}(\rho^{-1})=0. \tag{3.19}$$

If we consider the equation

$$e^{\rho\omega_1}e^{D_1-D_2}-e^{\rho\omega_2}e^{D_1+D_2}=0,$$

then, since $\omega_1=i$ and $\omega_2=-i$, it is just

$$e^{2i\rho-2D_2}=1$$

with solutions

$$\tilde{\mu}_k=i\rho_k=D_2+k\pi i,\quad k=1,2,\ldots, \tag{3.20}$$

where D_2 is defined by (3.9). Using (3.20) and the Rouché's theorem, the solutions of (3.19) will satisfy

$$\mu_k=D_2+k\pi i+\mathcal{O}(k^{-1}),\quad k=N,N+1,\ldots, \tag{3.21}$$

where N is a sufficiently large positive integer. Since $\mu=h\lambda$, we have

$$\lambda_k=\frac{1}{h}\mu_k=\frac{1}{h}(D_2+k\pi i)+\mathcal{O}(k^{-1}),\quad k\geqslant N. \tag{3.22}$$

Also, for large enough k, λ_k is simple because problem (2.15) is strongly regular [12, pp. 64–74]. □

Remark 3.1. We remark here to explain the interplay between the strong regularity and the simplicity of the eigenvalues for problem (2.15) (or equivalently (2.13)). From the characteristic Eq. (2.15) (or (2.13)) and the structure of the corresponding Green's function for the inverse of the associated ordinary differential operator [12, pp. 34–37], the multiplicity of each $\lambda\in\sigma(\mathscr{A})$ with large enough modulus, as a pole of the resolvent operator $R(\lambda,\mathscr{A})$, is less than or equal to the order of λ

as a zero for the entire function $\Delta(\rho)$ with $\lambda := \mathrm{i}\rho/h$. On the other hand, since problem (2.15) is strongly regular, it is easy to verify that λ is geometrically simple when modulus of λ is large and the zeros of $\Delta(\rho)=0$ are simple when their moduli are large. So eigenvalues of problem (2.15) with sufficiently large moduli are algebraically simple because: $m_{\mathrm{a}} \leqslant p \cdot m_{\mathrm{g}}$ [11, pp. 148], where p denotes the order of the pole of the resolvent operator and $m_{\mathrm{a}}, m_{\mathrm{g}}$ denote the algebraic and geometric multiplicities, respectively.

All the above discussions can be summarized as follows.

Theorem 3.3. *Let $\mathscr{A}+\mathscr{B}$ be defined as in* (2.3) *and* (2.4). *Then each $\lambda \in \sigma(\mathscr{A}+\mathscr{B})$ is simple when $|\lambda|$ is large enough, and has an asymptotic expression given by* (3.16).

4. Exponential stability of the system

In this section, we investigate the Riesz basis property and the stability of system (2.5). The completeness of the generalized eigenfunctions of $\mathscr{A}$ comes from a classical completeness result due to V.M. Keldysh [4, pp. 170, Theorem 4.1] below.

Lemma 4.1. *Let K be a compact self-adjoint operator in a Hilbert space* **H** *with* $\ker K=\{0\}$ *and eigenvalues $\lambda_j(K)$, $j=1,2,\ldots,\infty$. Assume that*

$$\sum_{j=1}^{\infty} |\lambda_j(K)|^r < \infty$$

for some $r \geqslant 1$, and let S be a compact operator such that $I+S$ is invertible. Then the system of generalized eigenfunctions of the operator

$$A := K(I+S)$$

is complete in **H**.

Theorem 4.1. *Let $\mathscr{A}+\mathscr{B}$ be defined as in* (2.3) *and* (2.4). *Then system* (2.5) *is complete in the sense that the generalized eigenfunctions of $\mathscr{A}+\mathscr{B}$ are complete in Hilbert space $\mathscr{H}$.*

Proof. Since $\mathscr{A}$ is a skew-adjoint operator with compact resolvents and $0, \infty \in \rho(\mathscr{A})$, so $(\mathrm{i}\mathscr{A})^{-1}$ is a compact self-adjoint operator with $\ker(\mathrm{i}\mathscr{A})^{-1}=\{0\}$. Also, by (3.16) (where $c_0(t) \equiv 0$), we have $\{\lambda_k((\mathrm{i}\mathscr{A})^{-1})\}_{k=1}^{\infty} \in l^2$. Since

$$\begin{aligned}(\mathrm{i}(\mathscr{A}+\mathscr{B}))^{-1} &= (\mathrm{i}\mathscr{A})^{-1}(I+\mathscr{B}\mathscr{A}^{-1})^{-1}\\ &= (\mathrm{i}\mathscr{A})^{-1}(I-\mathscr{B}\mathscr{A}^{-1}(I+\mathscr{B}\mathscr{A}^{-1})^{-1}),\end{aligned}$$

$\mathscr{B}\mathscr{A}^{-1}$ and $\mathscr{B}\mathscr{A}^{-1}(I+\mathscr{B}\mathscr{A}^{-1})^{-1}$ are compact and $I-\mathscr{B}\mathscr{A}^{-1}(I+\mathscr{B}\mathscr{A}^{-1})^{-1}$ is invertible, so the proof follows from Lemma 4.1. □

We also need the following Theorem 4.2 from [21].

Theorem 4.2. *Let X be a separable Hilbert space, and $\mathscr{A}$ be the generator of a C_0-semigroup $T(t)$ on X. Suppose that the following conditions hold:*

1. *$\sigma(\mathscr{A})=\sigma_1(\mathscr{A})\cup\sigma_2(\mathscr{A})$ and $\sigma_2(\mathscr{A})=\{\lambda_k\}_{k=1}^{\infty}$ consists of only isolated eigenvalues of finite algebraic multiplicity;*
2. *for $m_a(\lambda_k) := \dim E(\lambda_k,\mathscr{A})X$, where $E(\lambda_k,\mathscr{A})$ denotes the Riesz-projection associated with λ_k, it has $\sup_{k\geqslant 1} m_a(\lambda_k) < \infty$;*
3. *there is a constant α such that*

$$\sup\{\operatorname{Re}\lambda \mid \lambda \in \sigma_1(\mathscr{A})\} \leqslant \alpha \leqslant \inf\{\operatorname{Re}\lambda \mid \lambda \in \sigma_2(\mathscr{A})\}$$

and $\inf_{n\neq m}|\lambda_n-\lambda_m| > 0$.

Then the following assertions hold:

(i) *There exist two $T(t)$-invariant closed subspaces X_1 and X_2 such that $\sigma(\mathscr{A}|_{X_1})=\sigma_1(\mathscr{A})$, $\sigma(\mathscr{A}|_{X_2})=\sigma_2(\mathscr{A})$, and $\{E(\lambda_k,\mathscr{A})X_2\}_{k=1}^{\infty}$ forms a Riesz basis of subspaces for X_2, i.e., $\forall\, x_2 \in X_2$,*

$$x_2 = \sum_{k=1}^{\infty} E(\lambda_k,\mathscr{A})x_2.$$

(ii) *There exist two positive constants C_1, C_2 independent of k and x_2 such that*

$$C_1 \sum_{k=1}^{\infty} \|E(\lambda_k,\mathscr{A})x_2\|^2 \leqslant \left\| \sum_{k=1}^{\infty} E(\lambda_k,\mathscr{A})x_2 \right\|^2 \leqslant C_2 \sum_{k=1}^{\infty} \|E(\lambda_k,\mathscr{A})x_2\|^2.$$

Furthermore,

$$X = \overline{X_1 \oplus X_2}.$$

(iii) *If $\sup_{k\geqslant 1}\|E(\lambda_k,\mathscr{A})\| < \infty$, then $D(\mathscr{A}) \subset X_1 \oplus X_2 \subset X$.*

(iv) *X can decompose into the topological direct sum $X = X_1 \oplus X_2$ if and only if*

$$\sup_{n\geqslant 1} \left\| \sum_{k=1}^{n} E(\lambda_k,\mathscr{A}) \right\| < \infty.$$

Combining Theorems 4.1, 4.2, 3.2 and 3.3, we obtain the first result of this paper.

Theorem 4.3. *System* (2.5) *is a Riesz spectral system* (*in the sense that its generalized eigenfunctions form a Riesz basis in* $\mathscr{H}$). *Thus, the spectrum determined growth condition holds, that is, $s(\mathscr{A}+\mathscr{B})=\omega(\mathscr{A}+\mathscr{B})$, with $s(\mathscr{A}+\mathscr{B}) := \sup\{\operatorname{Re}\lambda \mid \lambda \in \sigma(\mathscr{A}+\mathscr{B})\}$ being the spectral bound of $\mathscr{A}+\mathscr{B}$ and $\omega(\mathscr{A}+\mathscr{B})$ being the growth bound of the semigroup $\mathrm{e}^{(\mathscr{A}+\mathscr{B})t}$.*

Proof. For system (2.5), from Theorems 3.2 and 3.3, we may take $\sigma_2(\mathscr{A}+\mathscr{B})=\sigma(\mathscr{A}+\mathscr{B})$, $\sigma_1(\mathscr{A}+\mathscr{B})=\{\infty\}$, then it is easy to see that conditions (2) and (3) in Theorem 4.2 are true. Finally, Theorem 4.1 implies that $X_1 = \{0\}$. Therefore, the first assertion of Theorem 4.2 says that there is a sequence of generalized eigenvectors of $\mathscr{A} + \mathscr{B}$ that forms a Riesz basis in $\mathscr{H}$. Accordingly, the spectrum determined growth condition can be obtained by a direct consequence of the Riesz basis property and the simplicity of the high eigen-frequencies of $\mathscr{A} + \mathscr{B}$. The proof is then complete. □

We are now ready to see how the damping coefficient $a(x)$ affects the asymptotic decay rate. From (3.18), we see that the high eigen-frequency components of the system will decay asymptotically at the rate of

$$-\frac{1}{2h}\int_0^1 \frac{a(x)}{EI(x)}\left(\frac{I_\rho(x)}{EI(x)}\right)^{-1/2} \mathrm{d}x.$$

We also note that under condition (1.2), Theorems 3.2 and 4.3 imply that for each $\varepsilon > 0$ there are at most finitely many eigenvalues on the right of the line

$$\operatorname{Re}\lambda = -\frac{1}{2h}\int_0^1 \frac{a(x)}{EI(x)}\left(\frac{I_\rho(x)}{EI(x)}\right)^{-1/2} \mathrm{d}x + \varepsilon.$$

We split our discussion into two separate cases according to the sign of $a(x)$.

Case 1: If $a(x) \geqslant 0$ and $a(x)|_\Omega > 0$ for $x \in [0, 1]$, where Ω is some measurable subset of $[0, 1]$ with positive measure, we claim that $\mathscr{A} + \mathscr{B}$ will be dissipative in $\mathscr{H}$. In that regard, we need only show that there is no eigenvalue on the imaginary axis. Clearly 0 is not an eigenvalue of $\mathscr{A} + \mathscr{B}$. Let $\lambda := \mathrm{i}\tau$, $\tau \in \mathbb{R}$, be an eigenvalue of $\mathscr{A} + \mathscr{B}$ and let $\Phi := [\phi, \lambda\phi]$ be the corresponding eigenfunction. Then we have

$$0 \equiv \operatorname{Re}\langle(\mathscr{A} + \mathscr{B})\Phi, \Phi\rangle_{\mathscr{H}} = -|\lambda|^2 \int_0^1 a(x)|\phi'(x)|^2\,\mathrm{d}x. \quad (4.1)$$

Since $a(x)$ and $\phi'(x)$ are continuous functions, we have $\phi'(x) \equiv 0$ for $x \in \Omega$ and thus $\phi'(x) \equiv 0$ for all $x \in [0, 1]$ (see [6]). From $\phi(0) = 0$, it follows that $\phi(x) \equiv 0$ for all $x \in [0, 1]$. So $\Phi = 0$, which is a contradiction. So there is no eigenvalues on the imaginary axis and Theorems 4.3 and 3.2 ensure that system (2.5) is exponentially stable.

Case 2: Assume that $a(x)$ can change its sign on $[0, 1]$ and satisfies condition (1.2). Let

$$a_+(x) := \max\{a(x), 0\}, \quad a_-(x) := \max\{-a(x), 0\},$$

and let $\mathscr{B}_\pm$ be the corresponding damping operators with respect to $a_\pm(x)$, respectively. Then it is easy to see that $a_+(x)$ satisfies the assumptions of Case 1 and that $\mathscr{B}_-$ is a self-adjoint operator with $\|\mathscr{B}_-\| \leqslant \max_{x\in[0,1]}\{a_-(x)/I_\rho(x)\}$. Decompose $\mathscr{A} + \mathscr{B}$ into

$$\mathscr{A} + \mathscr{B} = \mathscr{A} + \mathscr{B}_+ - \mathscr{B}_-.$$

If

$$\max_{x\in[0,1]}\left\{\frac{a_-(x)}{I_\rho(x)}\right\} < |s(\mathscr{A} + \mathscr{B}_+)|, \quad (4.2)$$

then the perturbation theory of contractive semigroups (see [13]) implies that $\lambda \in \rho(\mathscr{A}+\mathscr{B})$ whenever $\operatorname{Re}\lambda > s(\mathscr{A}+\mathscr{B}_+) + \|\mathscr{B}_-\|$. From Theorem 4.3 again, we see that

$$\omega(\mathscr{A} + \mathscr{B}) = s(\mathscr{A} + \mathscr{B}) \leqslant s(\mathscr{A} + \mathscr{B}_+) + \|\mathscr{B}_-\| < 0$$

and thus system (2.5) is exponentially stable.

We summarize the above discussions into the result below.

Theorem 4.4. *Suppose that conditions* (1.2) *and* (1.3) *hold and* $a(x)|_\Omega > 0$ *is continuously differentiable on* $[0, 1]$, *where* Ω *is some measurable subset of* $[0, 1]$ *with positive measure.*

(1) *If* $a(x) \geqslant 0$, *then system* (2.5) *is exponentially stable.*
(2) *If* $a(x)$ *is indefinite*, *then system* (2.5) *is exponentially stable whenever condition* (4.2) *is satisfied.*

Remark 4.1. Although computing $|s(\mathscr{A} + \mathscr{B}_+)|$ in (4.2) is not very straightforward, condition (4.2) does provide a condition on how negative the damping term could be while the exponential stability can still hold. Numerical algorithms could also be used in estimating $|s(\mathscr{A} + \mathscr{B}_+)|$.

Remark 4.2. Condition (1.2) is very valuable to system (1.1) with infinite damping because it can supply a calculable quantity that reflects the relation between positive and negative parts. Here we shall give an example for system (1.1) with a prescribed function $a(x)$ and it will also have the same arguments with other functions on $a(x)$. For simplicity, we assume that $\rho(x) = I_\rho(x) = EI(x) \equiv 1$. Let $\delta > 0$ and

$$a(x) = \begin{cases} -\delta(x - \frac{1}{4})^2, & x \in [0, \frac{1}{4}], \\ (x - \frac{1}{4})^2, & x \in (\frac{1}{4}, 1]. \end{cases}$$

Then $a(x)$ is indefinite in $[0, 1]$. To obtain the exponential stability of the system, from condition (1.2) and the fact that

$$\int_0^1 a(x)\,\mathrm{d}x = -\delta\int_0^{1/4}\left(x - \frac{1}{4}\right)^2 \mathrm{d}x + \int_{1/4}^1 \left(x - \frac{1}{4}\right)^2 \mathrm{d}x = \frac{9}{64} - \frac{\delta}{192},$$

we see that $\delta < 27$. This will ensure that the high frequencies locate in the left half plane. Furthermore, since $\max_{x\in[0,1]} a_-(x) = \delta/16$, where

$$a_-(x) := \max\{-a(x), 0\} = \begin{cases} \delta(x - \frac{1}{4})^2, & x \in [0, \frac{1}{4}], \\ 0, & x \in (\frac{1}{4}, 1], \end{cases}$$

and by virtual of (4.2) and Theorem 4.4, there exists $0 < \delta_0 < 27$, such that for each $\delta \in (0, \delta_0)$, system (1.1) is exponentially stable.

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