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Applied Mathematics Letters 18 (2005) 127–134
Applied Mathematics Letters
www.elsevier.com/locate/aml

Spectral analysis and system of fundamental solutions for Timoshenko beams☆

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Received 1 June 2003; received in revised form 1 August 2004; accepted 1 September 2004

Abstract

We have found a unified method to analyse Timoshenko beams under various boundary conditions that occurred in practice. Explicit asymptotic expressions for the spectrum are obtained. Our method is very simple but effective because explicit formulas are obtained for the system of fundamental solutions, which are very useful for other purposes such as stability analysis. The eigenfunctions are also shown to form an orthogonal basis.

Keywords: Timoshenko beam; Fundamental solutions; Asymptotic spectral distribution

1. Introduction

Many mechanical systems, such as spacecraft or robot arms with flexible links, can be modeled as coupled elastic and rigid parts. Many future space applications, such as the space station, rely on lightweight materials and high performance control systems for the requirements of the high precision pointing and tracking. To achieve precision demands for such systems, one has to take the effect of

☆ The authors are grateful to the anonymous referees for their helpful suggestions and comments. The research of the third author was supported by the National Science Foundation grant NSFC-60474017. The research of the fourth author was supported by an RGC grant of code HKU 7133/02P.
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0893-9659/$ - see front matter
doi:10.1016/j.aml.2004.09.001

flexible parts into account. The Timoshenko beam model is widely employed to describe such systems; see [1–3] for example. Two coupled partial differential equations arise in such a model,

$$\begin{cases} \rho w_{tt}(x,t) - K(w_{xx}(x,t) - \varphi_x(x,t)) = 0, & 0 < x < \ell, t > 0, \\ I_\rho \varphi_{tt}(x,t) - EI\varphi_{xx}(x,t) - K(w_x(x,t) - \varphi(x,t)) = 0, & 0 < x < \ell, t > 0 \end{cases} \tag{1.1}$$

where ℓ is the length of a uniform beam, ρ is the mass per unit length, $w(x,t)$ is the deflection of the beam from its equilibrium and $\varphi(x,t)$ is the total rotatory angle of the beam at the spatial position x and time t; see [1,4]. Constants I_ρ and EI are the mass moment of inertia and rigidity coefficients of the cross-section, respectively, and K is the shear modulus of elasticity. Various boundary conditions will be applied to Eq. (1.1) in applications (see [5]) and our aim in this paper is to treat those that occur frequently in practice by a unified method. The boundary conditions that we will consider are

(B_1) free–free: $\begin{cases} K(w_x(0,t) - \varphi(0,t)) = 0, & EI\varphi_x(0,t) = 0, \\ K(w_x(\ell,t) - \varphi(\ell,t)) = 0, & EI\varphi_x(\ell,t) = 0, \end{cases}$

(B_2) built in–free: $\begin{cases} w(0,t) = 0, & \varphi(0,t) = 0, \\ K(w_x(\ell,t) - \varphi(\ell,t)) = 0, & EI\varphi_x(\ell,t) = 0, \end{cases}$

(B_3) hinged–hinged: $\begin{cases} w(0,t) = 0, & EI\varphi_x(0,t) = 0, \\ w(\ell,t) = 0, & EI\varphi_x(\ell,t) = 0, \end{cases}$

(B_4) built in–built in: $\begin{cases} w(0,t) = 0, & \varphi(0,t) = 0, \\ w(\ell,t) = 0, & \varphi(\ell,t) = 0, \end{cases}$

(B_5) built in–hinged: $\begin{cases} w(0,t) = 0, & \varphi(0,t) = 0, \\ w(\ell,t) = 0, & EI\varphi_x(\ell,t) = 0. \end{cases}$

After making a standard separation of variable argument, one obtains the Timoshenko eigenvalue equations

$$\begin{cases} \rho\lambda^2 w(x) - K(w''(x) - \varphi'(x)) = 0, & 0 < x < \ell, \\ I_\rho \lambda^2 \varphi(x) - EI\varphi''(x) - K(w'(x) - \varphi(x)) = 0, & 0 < x < \ell \end{cases} \tag{1.2}$$

where prime ($'$) denotes the differentiation with respect to x, λ is the eigenvalue parameter and the x-dependent portions are still denoted by w and φ.

The main purpose of the present paper is to obtain explicit formulas for a fundamental solution of (1.2) in Section 2. We also obtain explicit representations for the eigenfrequencies as well as the eigenfunctions in Section 3.

2. Fundamental solutions for Timoshenko equations

The solution formula of the Timoshenko eigenvalue Eq. (1.2) can be obtained via some lengthy, but elementary, calculations. All verifications can be done simply by direct substitutions and so the details are skipped. Here is the main theorem of this paper.

Theorem 2.1. *Let $\lambda \in \mathbb{C}$. Let $(w(x,\lambda), \varphi(x,\lambda))$ be a solution pair of the Timoshenko eigenvalue Eq. (1.2) with eigenvalue parameter λ. Set*

$$a := \frac{\rho}{K}\lambda^2, \qquad b := \frac{I_\rho}{EI}\lambda^2 + \frac{K}{EI}, \qquad c := -\frac{K}{EI} \tag{2.1}$$

and denote by μ_1 and μ_2 the two roots of the quadratic equation

$$\mu^2 - (a+b+c)\mu + ab = 0. \tag{2.2}$$

Assume that $\mu_1 \neq \mu_2$, then w and φ are given by

$$w(x,\lambda) = w(0)w_1(x,\lambda) + \varphi(0)w_2(x,\lambda) + w'(0)w_3(x,\lambda) + \varphi'(0)w_4(x,\lambda)$$

and

$$\varphi(x,\lambda) = w(0)\varphi_1(x,\lambda) + \varphi(0)\varphi_2(x,\lambda) + w'(0)\varphi_3(x,\lambda) + \varphi'(0)\varphi_4(x,\lambda)$$

where the fundamental solutions are given by

$$\begin{aligned}
w_1(x,\lambda) &:= \frac{1}{\mu_1-\mu_2}\left((\mu_1-a)\cosh\sqrt{\mu_2}x - (\mu_2-a)\cosh\sqrt{\mu_1}x\right),\\
\varphi_1(x,\lambda) &:= \frac{ac}{\mu_1-\mu_2}\left(\sqrt{\mu_1}^{-1}\sinh\sqrt{\mu_1}x - \sqrt{\mu_2}^{-1}\sinh\sqrt{\mu_2}x\right),\\
w_2(x,\lambda) &:= \frac{b}{\mu_1-\mu_2}\left(\sqrt{\mu_1}^{-1}\sinh\sqrt{\mu_1}x - \sqrt{\mu_2}^{-1}\sinh\sqrt{\mu_2}x\right),\\
\varphi_2(x,\lambda) &:= \frac{1}{\mu_1-\mu_2}\left((\mu_1-b)\cosh\sqrt{\mu_2}x - (\mu_2-b)\cosh\sqrt{\mu_1}x\right),\\
w_3(x,\lambda) &:= \frac{1}{\mu_1-\mu_2}\left((\mu_1-b)\sqrt{\mu_1}^{-1}\sinh\sqrt{\mu_1}x - (\mu_2-b)\sqrt{\mu_2}^{-1}\sinh\sqrt{\mu_2}x\right),\\
\varphi_3(x,\lambda) &:= \frac{c}{\mu_1-\mu_2}\left(\cosh\sqrt{\mu_1}x - \cosh\sqrt{\mu_2}x\right),\\
w_4(x,\lambda) &:= \frac{1}{\mu_1-\mu_2}\left(\cosh\sqrt{\mu_1}x - \cosh\sqrt{\mu_2}x\right),\\
\varphi_4(x,\lambda) &:= \frac{1}{\mu_1-\mu_2}\left((\mu_1-a)\sqrt{\mu_1}^{-1}\sinh\sqrt{\mu_1}x - (\mu_2-a)\sqrt{\mu_2}^{-1}\sinh\sqrt{\mu_2}x\right).
\end{aligned}$$

Remark 2.1. In the case of double roots (that means $\mu_1 = \mu_2$), one can derive the corresponding solution expressions by taking the limit $\mu_1 \to \mu_2$ in the above expressions. The proof of Theorem 2.1 can also be used to analyse asymptotic behavior of the eigenfrequencies as in Theorem 3.2.

Since the functions $(w_k(x,\lambda), \varphi_k(x,\lambda))$, $k = 1, 2, 3, 4$, form a fundamental solution set for (1.2) that satisfy

$$\begin{aligned}
\left(w_1(0,\lambda), \varphi_1(0,\lambda), w_1'(0,\lambda), \varphi_1'(0,\lambda)\right) &= (1,0,0,0),\\
\left(w_2(0,\lambda), \varphi_2(0,\lambda), w_2'(0,\lambda), \varphi_2'(0,\lambda)\right) &= (0,1,0,0),\\
\left(w_3(0,\lambda), \varphi_3(0,\lambda), w_3'(0,\lambda), \varphi_3'(0,\lambda)\right) &= (0,0,1,0),\\
\left(w_4(0,\lambda), \varphi_4(0,\lambda), w_4'(0,\lambda), \varphi_4'(0,\lambda)\right) &= (0,0,0,1)
\end{aligned}$$

so the corresponding solution expressions for the inhomogeneous equation can be obtained immediately. These solution expressions are very useful for the study of feedback stabilization (see for instance [6,7]).

Theorem 2.2. *Let $\lambda \in \mathbb{C}$. Assume that (w, φ) is a solution pair of the inhomogeneous Timoshenko eigenvalue equations with parameter λ:*

$$\begin{cases} \rho\lambda^2 w(x) - K(w''(x) - \varphi'(x)) = \rho f_1(x), & 0 < x < \ell, \\ I_\rho \lambda^2 \varphi(x) - EI\varphi''(x) - K(w'(x) - \varphi(x)) = I_\rho f_2(x), & 0 < x < \ell. \end{cases} \tag{2.3}$$

If $\mu_1 \neq \mu_2$, then (w, φ) are given by

$$\begin{aligned} w(x) = {} & w(0)w_1(x,\lambda) + \varphi(0)w_2(x,\lambda) + w'(0)w_3(x,\lambda) + \varphi'(0)w_4(x,\lambda) \\ & + \rho_1^2 \int_0^x w_3(x-s,\lambda) f_1(s)\mathrm{d}s + \rho_2^2 \int_0^x w_4(x-s,\lambda) f_2(s)\mathrm{d}s \end{aligned}$$

and

$$\begin{aligned} \varphi(x) = {} & w(0)\varphi_1(x,\lambda) + \varphi(0)\varphi_2(x,\lambda) + w'(0)\varphi_3(x,\lambda) + \varphi'(0)\varphi_4(x,\lambda) \\ & + \rho_1^2 \int_0^x \varphi_3(x-s,\lambda) f_1(s)\mathrm{d}s + \rho_2^2 \int_0^x \varphi_4(x-s,\lambda) f_2(s)\mathrm{d}s \end{aligned}$$

where $\rho_1 := \sqrt{\frac{\rho}{K}}$, $\rho_2 := \sqrt{\frac{I_\rho}{EI}}$ and w_i, φ_i, $i = 1, 2, 3, 4$ are defined in Theorem 2.1.

Proof. By direct substitutions, one can verify that the integral terms form a particular solution to (2.3). □

3. Spectral analysis of Timoshenko beams under various boundary conditions

In this section, we will perform an asymptotic analysis for the eigenfrequencies of Timoshenko beams under various boundary conditions. We begin with defining the state spaces.

Define the space $\mathcal{H} := H^1(0,\ell) \times L^2(0,\ell) \times H^1(0,\ell) \times L^2(0,\ell)$ with $H^k(0,\ell)$ $(k = 1, 2)$ being the usual Sobolev space of order k. For $Y_1 = [w_1, z_1, \varphi_1, \psi_1]^{\mathrm{T}}$, $Y_2 = [w_2, z_2, \varphi_2, \psi_2]^{\mathrm{T}} \in \mathcal{H}$, the inner product in $\mathcal{H}$ is defined by

$$\begin{aligned} \langle Y_1, Y_2 \rangle := {} & \int_0^\ell K(w_1' - \varphi_1)\overline{(w_2' - \varphi_2)}\mathrm{d}x + \int_0^\ell \rho z_1 \overline{z_2}\mathrm{d}x + \int_0^\ell EI\varphi_1'\overline{\varphi_2'}\mathrm{d}x + \int_0^\ell I_\rho \psi_1 \overline{\psi_2}\mathrm{d}x \\ & + K w_1(0)\overline{w_2(0)} + EI\varphi_1(0)\overline{\varphi_2(0)} \end{aligned}$$

and hence it is easy to see that $\mathcal{H}$ is a Hilbert space. Define subspaces of $\mathcal{H}$ by

$$\mathcal{H}_2 := \{Y = [w, z, \varphi, \psi]^{\mathrm{T}} \in \mathcal{H} \mid w(0) = 0, \varphi(0) = 0\},$$
$$\mathcal{H}_3 := \{Y = [w, z, \varphi, \psi]^{\mathrm{T}} \in \mathcal{H} \mid w(0) = 0\}$$

and we have

$$\mathcal{H}_3 = \mathcal{H}_2 + \{\xi[0, 0, 1, 0]^{\mathrm{T}} \mid \xi \in \mathbb{C}\}$$

and

$$\mathcal{H} = \mathcal{H}_2 + \{\xi[1, 0, 0, 0]^{\mathrm{T}} + \zeta[0, 0, 1, 0]^{\mathrm{T}} \mid \xi, \zeta \in \mathbb{C}\}.$$

We now consider the Timoshenko beam Eq. (1.1) with each one of the following boundary conditions: (B_1) free–free; (B_2) built in–free; (B_3) hinged–hinged; (B_4) built in–built in and (B_5) built in–hinged (for their physical significance, please refer to [5]).

Set $\mathcal{H}_1 := \mathcal{H}, \mathcal{H}_4 := \mathcal{H}_2, \mathcal{H}_5 := \mathcal{H}_2$; all are subspaces in $\mathcal{H}$. On them we define the linear operator $\mathcal{A}_j$ by

$$\mathcal{A}_j \begin{pmatrix} w \\ z \\ \varphi \\ \psi \end{pmatrix} = \begin{pmatrix} z \\ \frac{K}{\rho}(w'' - \varphi') \\ \psi \\ \frac{EI}{I_\rho}\varphi'' + \frac{K}{I_\rho}(w' - \varphi) \end{pmatrix}, \qquad \forall \begin{pmatrix} w \\ z \\ \varphi \\ \psi \end{pmatrix} \in \mathcal{D}(\mathcal{A}_j), \qquad j = 1, 2, 3, 4, 5 \tag{3.1}$$

with domains

$$\mathcal{D}(\mathcal{A}_1) = \{Y = [w, z, \varphi, \psi]^{\mathrm{T}} \in \mathcal{H}_1 \mid w, \varphi \in H^2(0, \ell),\ z, \psi \in H^1(0, \ell), \\ K(w'(x) - \varphi(x))|_{x=0,\ell} = 0, EI\varphi'(x)|_{x=0,\ell} = 0\}, \tag{3.2}$$

$$\mathcal{D}(\mathcal{A}_2) = \{Y = [w, z, \varphi, \psi]^{\mathrm{T}} \in \mathcal{H}_2 \mid w, \varphi \in H^2(0, \ell),\ z, \psi \in H^1(0, \ell), \\ K(w'(\ell) - \varphi(\ell)) = 0, EI\varphi'(\ell) = 0\}, \tag{3.3}$$

$$\mathcal{D}(\mathcal{A}_3) = \{Y = [w, z, \varphi, \psi]^{\mathrm{T}} \in \mathcal{H}_3 \mid w, \varphi \in H^2(0, \ell),\ z, \psi \in H^1(0, \ell), \\ z(\ell) = w(\ell) = 0, EI\varphi'(x)|_{x=0,\ell} = 0\}, \tag{3.4}$$

$$\mathcal{D}(\mathcal{A}_4) = \{Y = [w, z, \varphi, \psi]^{\mathrm{T}} \in \mathcal{H}_4 \mid w, \varphi \in H^2(0, \ell),\ z, \psi \in H^1(0, \ell), \\ z(\ell) = w(\ell) = 0, \varphi(\ell) = \psi(\ell) = 0\}, \tag{3.5}$$

$$\mathcal{D}(\mathcal{A}_5) = \{Y = [w, z, \varphi, \psi]^{\mathrm{T}} \in \mathcal{H}_5 \mid w, \varphi \in H^2(0, \ell),\ z, \psi \in H^1(0, \ell), \\ z(\ell) = w(\ell) = 0, EI\varphi'(\ell) = 0\}. \tag{3.6}$$

It is clear that each $\mathcal{A}_j$ with domain $\mathcal{D}(\mathcal{A}_j)$ is the Timoshenko operator corresponding to boundary condition (B_j), and Eq. (1.1) with boundary condition (B_j) becomes the following evolution equation in $\mathcal{H}_j$:

$$\frac{\mathrm{d}Y(t)}{\mathrm{d}t} = \mathcal{A}_j Y(t), \qquad t > 0 \tag{3.7}$$

where $Y = [w(\cdot, t), \dot{w}(\cdot, t), \varphi(\cdot, t), \dot{\varphi}(\cdot, t)]^{\mathrm{T}}$. Under this evolution framework, we immediately have the following results, whose proofs are straightforward.

Theorem 3.1. *Let $\mathcal{A}_j$ be defined as above. Then*

(1) $\lambda = 0$ *is a double eigenvalue of* $\mathcal{A}_1$ *and the eigenvectors are*

$$Z_1 = [0, 0, 1, 0]^{\mathrm{T}}, \qquad Z_2 = [x, 0, 1, 0]^{\mathrm{T}};$$

$\lambda = \pm \mathrm{i}\sqrt{K/I_\rho}$ *are eigenvalues of* $\mathcal{A}_3$ *and the corresponding eigenvectors are*

$$\widehat{Z}_1 = [0, 0, 1, \mathrm{i}\sqrt{K/I_\rho}]^{\mathrm{T}}, \qquad \widehat{Z}_2 = [0, 0, 1, -\mathrm{i}\sqrt{K/I_\rho}]^{\mathrm{T}}$$

and clearly

$$[0, 0, 1, 0]^{\mathrm{T}} = \frac{1}{2}\widehat{Z}_1 + \frac{1}{2}\widehat{Z}_2;$$

(2) $\mathcal{A}_j$ *is skew adjoint on an invariant subspace with co-dimensional at most* 2*, and the resolvent of* $\mathcal{A}_j$ *is compact in* $\mathcal{H}_j$*;*

(3) $\sigma(\mathcal{A}_j) = \sigma_p(\mathcal{A}_j)$ *which consists of isolated eigenvalues with finite multiplicity of* $\mathcal{A}_j$*;*

(4) $\sigma(\mathcal{A}_j)$ *is symmetric with respect to the real axis and*

$$\sigma(\mathcal{A}_j) = \{\mathrm{i}\eta_n \mid n = (0), \pm 1, \pm 2, \ldots, \eta_{-n} = -\eta_n\},$$

with real $\eta_n > 0$ *for all* $n > 0$ *and* $\lim_{n\to\infty} \eta_n = \infty$;

(5) *all eigenvectors* $\{\Psi_n(x) \mid n = \pm 1, \pm 2, \ldots\}$ *of* $\mathcal{A}_j$ *form an orthogonal basis of* $\mathcal{H}_j$ *because of* (2), *and hence each* $\mathcal{A}_j$ *is an infinitesimal generator of a* C_0*-semigroup of linear operator and the corresponding solution can be represented as*

$$Y(x,t) = \sum_{n=-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}\eta_n t} \|\Psi_n\|^{-2} \langle Y(\cdot, 0), \Psi_n\rangle \Psi_n(x).$$

To consider the eigenvalue problem of $\mathcal{A}_j$, assume $\lambda \in \mathrm{i}\mathbb{R}$ such that the equation $(\lambda I - \mathcal{A}_j)Y = 0$ has a nonzero solution $Y = [w, \lambda w, \varphi, \lambda\varphi]^{\mathrm{T}} \in \mathcal{D}(\mathcal{A}_j)$. From Theorem 2.1, we have

$$Y = w(0)Y_1(\lambda) + \varphi(0)Y_2(\lambda) + w'(0)Y_3(\lambda) + \varphi'(0)Y_4(\lambda)$$

where

$$Y_k(\lambda) = [w_k(x,\lambda), \lambda w_k(x,\lambda), \varphi_k(x,\lambda), \lambda\varphi_k(x,\lambda)]^{\mathrm{T}}, \qquad k = 1, 2, 3, 4.$$

Without loss of generality, let $\rho_2 \neq \rho_1$. Substituting the solution formulas of Theorem 2.1 into the characteristic determinant of $\mathcal{A}_1$, which is given by

$$D_1(\lambda) = \begin{vmatrix} 0 & -1 & 1 \\ w_1'(\ell,\lambda) - \varphi_1(\ell,\lambda) & w_2'(\ell,\lambda) - \varphi_2(\ell,\lambda) & w_3'(\ell,\lambda) - \varphi_3(\ell,\lambda) \\ \varphi_1'(\ell,\lambda) & \varphi_2'(\ell,\lambda) & \varphi_3'(\ell,\lambda) \end{vmatrix} = 0, \tag{3.8}$$

of $\mathcal{A}_2$, which is given by

$$D_2(\lambda) = \begin{vmatrix} w_3'(\ell,\lambda) - \varphi_3(\ell,\lambda) & w_4'(\ell,\lambda) - \varphi_4(\ell,\lambda) \\ \varphi_3'(\ell,\lambda) & \varphi_4'(\ell,\lambda) \end{vmatrix} = 0, \tag{3.9}$$

of $\mathcal{A}_3$, which is given by

$$D_3(\lambda) = \begin{vmatrix} w_2(\ell,\lambda) & w_3(\ell,\lambda) \\ \varphi_2'(\ell,\lambda) & \varphi_3'(\ell,\lambda) \end{vmatrix} = 0, \tag{3.10}$$

of $\mathcal{A}_4$, which is given by

$$D_4(\lambda) = \begin{vmatrix} w_3(\ell,\lambda) & w_4(\ell,\lambda) \\ \varphi_3(\ell,\lambda) & \varphi_4(\ell,\lambda) \end{vmatrix} = 0, \tag{3.11}$$

of $\mathcal{A}_5$, which is given by

$$D_5(\lambda) = \begin{vmatrix} w_3(\ell,\lambda) & w_4(\ell,\lambda) \\ \varphi_3'(\ell,\lambda) & \varphi_4'(\ell,\lambda) \end{vmatrix} = 0, \tag{3.12}$$

we have the following asymptotic expressions (as $|\mathrm{Im}\,\lambda| \to \infty$)

$$D_1(\lambda) = \lambda^2 \rho_1\rho_2 \sinh(\rho_1\lambda\ell)\sinh(\rho_2\lambda\ell) + O(1),$$

$$D_2(\lambda) = \cosh(\rho_1\lambda\ell)\cosh(\rho_2\lambda\ell) + O\left(\frac{1}{\lambda^2}\right),$$

$$D_3(\lambda) = \sinh(\rho_1\lambda\ell)\sinh(\rho_2\lambda\ell) + O\left(\frac{1}{\lambda^2}\right),$$

$$D_4(\lambda) = \frac{1}{\lambda^2}[\sinh(\rho_1\lambda\ell)\sinh(\rho_2\lambda\ell)] + O\left(\frac{1}{\lambda^4}\right),$$
$$D_5(\lambda) = \frac{1}{\lambda}[\sinh(\rho_1\lambda\ell)\cosh(\rho_2\lambda\ell)] + O\left(\frac{1}{\lambda^3}\right).$$

Let

$$\zeta_n := \frac{n\pi\mathrm{i}}{\ell}, \qquad \xi_n := \frac{2(n+1)\pi\mathrm{i}}{2\ell}$$

and apply Rouché's theorem; we obtain the following result using the fact that $\lambda \in \sigma(\mathcal{A}_j)$ if and only if $D_j(\lambda) = 0$.

Theorem 3.2.

$$\sigma(\mathcal{A}_1) = \left\{\lambda_n^{(1)} = \frac{\zeta_n}{\rho_1} + \varepsilon_n^{(1)} \mid n \in \mathbb{Z}\right\} \cup \left\{\lambda_n^{(2)} = \frac{\zeta_n}{\rho_2} + \varepsilon_n^{(2)} \mid n \in \mathbb{Z}\right\} \cup \{0\},$$
$$\sigma(\mathcal{A}_2) = \left\{\lambda_n^{(1)} = \frac{\xi_n}{\rho_1} + \varepsilon_n^{(1)} \mid n \in \mathbb{Z}\right\} \cup \left\{\lambda_n^{(2)} = \frac{\xi_n}{\rho_2} + \varepsilon_n^{(2)} \mid n \in \mathbb{Z}\right\},$$
$$\sigma(\mathcal{A}_3) = \left\{\lambda_n^{(1)} = \frac{\zeta_n}{\rho_1} + \varepsilon_n^{(1)} \mid n \in \mathbb{Z}\right\} \cup \left\{\lambda_n^{(2)} = \frac{\zeta_n}{\rho_2} + \varepsilon_n^{(2)} \mid n \in \mathbb{Z}\right\},$$
$$\sigma(\mathcal{A}_4) = \left\{\lambda_n^{(1)} = \frac{\zeta_n}{\rho_1} + \varepsilon_n^{(1)} \mid n \in \mathbb{Z}\right\} \cup \left\{\lambda_n^{(2)} = \frac{\zeta_n}{\rho_2} + \varepsilon_n^{(2)} \mid n \in \mathbb{Z}\right\},$$
$$\sigma(\mathcal{A}_5) = \left\{\lambda_n^{(1)} = \frac{\zeta_n}{\rho_1} + \varepsilon_n^{(1)} \mid n \in \mathbb{Z}\right\} \cup \left\{\lambda_n^{(2)} = \frac{\xi_n}{\rho_2} + \varepsilon_n^{(2)} \mid n \in \mathbb{Z}\right\}$$

where $\varepsilon_n^{(j)} := O(\frac{1}{n})$ $(j = 1, 2)$ *for* $|n|$ *large enough.*

Remark 3.1. We would also like to note that our results in Theorem 3.2 include those in [8,9] in the constant coefficient case. Furthermore, our results in Theorems 2.1 and 2.2 can be used together with suitable boundary feedback controls to achieve feedback stabilities (see [6,7,10] for example).

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