Riesz basis property, exponential stability of variable coefficient Euler–Bernoulli beams with indefinite damping

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We study damped Euler–Bernoulli beams that have nonuniform thickness or density. These nonuniform features result in variable coefficient beam equations. We prove that despite the nonuniform features, the eigenfunctions of the beam form a Riesz basis and asymptotic behaviour of the beam system can be deduced without any restrictions on the sign of the damping. We also provide an answer to the frequently asked question on damping: ‘how much more positive than negative should the damping be without disrupting the exponential stability?’, and result in a criterion condition which ensures that the system is exponentially stable.

Keywords: variable coefficient; Euler–Bernoulli beam; Riesz basis property; exponential stability.

1. Introduction

We investigate variable coefficient damped Euler–Bernoulli beams in this paper. The variable coefficient feature arises frequently from beams that have nonuniform density or variable thickness (see Liew et al., 1995). A review of the literature quickly reveals many works on constant coefficient Euler–Bernoulli beams but relatively few on variable coefficient problems, for instance those of Guo (2002a,b) and Xu & Feng (2002), and they treat boundary feedback problems or free beam systems, which are different from our damped problem. Our approach is also different. In fact, our goal in this paper is not just tackling a particular problem but to exhibit a powerful machinery laid down by Naimark (1967) to handle a wide class of variable coefficient problems arising from beams. This methodology is so effective that it can be applied to a wide range of general cases including strings, Euler–Bernoulli, Rayleigh, Timoshenko, single or coupled beams. This is the first paper of a series that reports results obtained from this methodology.

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We find that a crucial step in the study of beams is to establish the Riesz basis property first, because then the spectrum-determined growth condition will hold and asymptotic behaviour as well as other properties can be deduced easily. The price to pay for this one-shot-does-it-all approach is a slightly complex setup due to the incorporation of Naimark's framework (1967). Also, our proof for the Riesz basis property is a little more complicated than conventional ones such as Guo (2002b) and Li (1978) because the method there cannot be used to treat variable coefficient problems—they used Bari's theorem for the high-frequency eigenfunctions and handle the low-frequency eigenfunctions in a case-by-case fashion, relying heavily on exact expressions of the characteristic equation and eigenfunctions, while our approach only requires asymptotic (not necessarily exact) expressions which emerge directly from the machinery provided in Naimark (1967). On the other hand, if one is only concerned about the exponential stability of the system, then one can bypass the Riesz basis property and follow a shorter route by using the conventional multiplier method together with the perturbation of compact operator such as in Komornik (1995). These methods are case-dependent due to the search for a suitable multiplier. However, our method will yield asymptotic behaviour of the system, eigenfrequencies, eigenfunctions and Riesz basis property all at once. Although the trade-off is a fairly complicated methodology, we believe that it is well-justified in view of its wider applications (see Wang et al., 2003, 2004). Finally, we would like to note that the Riesz basis property is equally important in many numerical applications (see for instance, Data et al., 2000, 2002; Mizusawa, 1993; Mutsuda & Sakiyama, 1988; Liew et al., 1995).

Our work shall make use of the following result from Wang (2003), which deals with the eigenvalue problem of beams in the form of an ordinary differential equation $Ly = \lambda y$ with $\lambda$-polynomial boundary conditions (see Shkalikov, 1986; Tretter, 1993).

To begin, we recall some notation and definitions. Let $L(y)$ be an ordinary differential operator of order $n = 2m \in \mathbb{N},$

$$L(y) = y^{(n)}(x) + \sum_{\nu = 1}^{n} f_{\nu}(x)y^{(n-\nu)}(x), \quad 0 < x < 1,$$

and let the boundary conditions defined at the two points $x = 0$ and $x = 1$ be

$$B_j(y) = \sum_{\nu = 0}^{k_j} (\alpha_{\nu j}y^{(k_j-\nu)}(0) + \beta_{\nu j}y^{(k_j-\nu)}(1)), \quad (1 \leq j \leq n),$$

where $k_j \in \mathbb{N}_0, 0 \leq k_j \leq n - 1$ and $\alpha_{\nu j}, \beta_{\nu j} \in \mathbb{C}, |\alpha_{\nu j}| + |\beta_{\nu j}| > 0.$

Suppose that the coefficient functions $f_{\nu}(x)$ $(1 \leq \nu \leq n)$ in (1.1) are sufficiently smooth in $(0, 1)$, and that the boundary conditions are normalized in the sense that $\kappa := \sum_{j=1}^{n} k_j$ is minimal with respect to all the equivalent boundary conditions (see Naimark, 1967).

Let $y_k(x, \rho) (k = 1, 2, \ldots, n)$ be the fundamental solutions for the equation

$$L(y) + \rho^n y + \rho^m \mu(x)y = 0, \quad \rho \in \mathbb{C}$$

with $\mu(x)$ being continuous in $[0, 1]$, and let $\omega_k (k = 1, 2, \ldots, n)$ be the $n$th roots of $\rho^n + 1 = 0.$ If we denote by $\Delta(\rho)$ the characteristic determinant of (1.3) with respect to (1.2)

$$\Delta(\rho) := \det[B_j(y_k(x, \rho))]_{j,k=1,2,\ldots,n},$$

then $\Delta(\rho)$ can be expressed asymptotically in the form (for $r \geq 1$)

$$\Delta(\rho) = \rho^r \sum_{j,k} e^{\mu_{j,k}} [F^j_k]_r,$$
whenever $\rho$ is large enough (see Shkalikov, 1986; Naimark, 1967). Here, $J_k$ is a $k$-element subset of \{1, 2, \ldots, n\}, $\mu_{J_k} = \sum_{j \in J_k} \omega_j$,

$$[F^\Delta]_k := F^\Delta_{0} + \rho^{-1} F^\Delta_{1} + \cdots + \rho^{-r+1} F^\Delta_{r-1} + O(\rho^{-r}),$$

and the sum runs over all possible selections of $J_k$.

**DEFINITION 1.1** The boundary problem (1.3) with (1.2) is said to be regular if the coefficients $F^\Delta_{0}$ in (1.4) are nonzero. Furthermore, the regular boundary problem (1.3) with (1.2) is said to be strongly regular if the zeros of $\Delta(\rho)$ are asymptotically simple and isolated one from another.

Let $W^m_2(0, 1)$ be the usual Sobolev space of order $m$ and let

$$V^m_E := [u(x) \in W^m_2(0, 1) \mid B_j(u) = 0, \ k_j < m].$$

Define a Hilbert space $H$ by

$$H := V^m_E[0, 1] \times L^2[0, 1]$$

with the norm $\|(f, g)\|_H^2 := \|f\|_{V^m_E}^2 + \|g\|_2^2$ and define the operator $A$ in $H$ by

$$\begin{align*}
A(f, g) := (g, -L(f) - \mu(x)g), \\
D(A) := \{F = (f, g) \in H \mid AF \in H, \ B_j(f) = 0, \ k_j \geq m\}. \\
(1.5)
\end{align*}$$

The following result was presented in Wang (2003).

**THEOREM 1.1** If the ordinary differential system with parameter $\lambda = \rho^m$

$$\begin{align*}
L(y, \lambda) &= L(y) + \lambda^2 y + \lambda \mu(x)y, \\
B_j(y) &= 0, \quad 1 \leq j \leq 2m
\end{align*}$$

has strongly regular boundary conditions, then the generalized eigenfunction system of $A$ forms a Riesz basis in the Hilbert space $H$.

In this paper, the problem that we shall consider is the following Euler–Bernoulli beam with indefinite viscous damping:

$$\begin{align*}
m(x)u_{tt}(t, x) + (EI(x)u_{xx}(t, x))_{xx} + \gamma(x)u_t(t, x) &= 0, \quad 0 < x < l, t > 0, \\
u(t, 0) &= u(t, l) = EI(0)u_{xx}(t, 0) = EI(l)u_{xx}(t, l) = 0, \quad t > 0, \\
u(0, x) &= u_0(x), \quad u_t(0, x) = u_1(x),
\end{align*}$$

(1.6)

where $l$ is the length of the beam, and $x$ stands for the spatial position and $t$ for time. Moreover, $EI(x)$ is the stiffness of the beam, $m(x)$ is the mass density, $u(t, x)$ denotes the transverse displacement, and $\gamma(x)$ is a continuous coefficient function of damping and satisfies the condition

$$\int_0^l \left( \frac{\gamma(x)}{m(x)} \right) \left( \frac{m(x)}{EI(x)} \right)^{1/4} dx > 0.$$  

(1.7)
Note that this condition will allow $\gamma$ to be indefinite in the interval $(0, l)$. If $\gamma > 0$, it can be proven that the energy of the system decays exponentially (see Theorem 3.2). Our question is, under our assumption, what condition on $\gamma$ can ensure that the system remains exponentially stable? We shall see later that condition (1.7) is necessary for the exponential stability of system (1.6).

In this paper, we always assume that $
abla (x), EI (x) \in C^4(0, l)$ and $EI (x), m (x) > 0$. (1.8)

With the assumptions (1.7) and (1.8), we shall prove that system (1.6) is a Riesz spectral system in the sense that the generalized eigenfunctions of the system form a Riesz basis on the suitable Hilbert space (see Curtain & Zwart, 1995). As a result, the asymptotic behaviour of the system can be obtained fairly easily. As we mentioned before, the Riesz basis property of the variable coefficient Euler–Bernoulli beam with boundary linear feedback controls have been discussed in Guo (2002a,b), in which the author establishes the Riesz basis property of the generalized eigenfunctions by using Bari’s method together with an analysis of the low-frequency modes. Here, we shall use a method entirely different from that used in Guo (2002a,b). Although it seems to be slightly more complicated, it can arrive at the conclusions all at once and be more readily applied to other general cases.

The methodology of our approach is quite simple: first we use the tools in Birkhoff (1908a,b) and Naimark (1967) to estimate the eigenvalues; then we use Theorem 1.1 to deduce the Riesz basis property and the exponential stability of the system.

The contents of the paper are arranged as follows. In Section 2, we convert system (1.6) into an abstract Cauchy problem in the Hilbert state space, and then discuss some basic properties of the system. We show that system (1.6) can be associated to a $C_0$-group, and the generator $A_\gamma$ of the $C_0$-group has compact resolvents. Furthermore, we will obtain an asymptotic expression for the eigenvalues. In Section 3, we will discuss the Riesz basis property of the eigenfunctions as well as the exponential stability of the system. Through a bounded invertible transform $T$, we establish the relationship between $A_\gamma$ and $A$ defined in Theorem 1.1 and obtain the Riesz basis property from the strong regularity of boundary conditions that has been verified in Section 2. Incidentally, we also obtain conditions for the exponential stability of the system for the indefinite damping.

2. Basic properties of system (1.6)

In this section, we put up some basic properties for system (1.6). For convenience, we may regard $l = 1$; otherwise, we can make a change of variable $x = lz$, and set

\[ U(t, z) := u(t, x), \quad 0 < z < 1 \]

and

\[ \hat{m}(z) := m(x)/l^2, \quad \hat{E}I(z) := EI(x)/l^2, \quad \hat{\gamma}(z) := \gamma(x)/l^2, \]

then system (1.6) is changed into

\[
\begin{cases}
\hat{m}(z)U_{tt}(t, z) + (\hat{E}I(z)U_{zz}(t, z))_{zz} + \hat{\gamma}(z)U_t(t, z) = 0, & 0 < z < 1, \; t > 0, \\
U(t, 0) = U(t, 1) = \hat{E}I(0)U_{zz}(t, 0) = \hat{E}I(1)U_{zz}(t, 1) = 0, & t > 0, \\
U(0, z) = u_0(lz), \quad U_t(0, z) = u_1(lz). &
\end{cases}
\] (2.1)
Considering (1.6) as a system in the Hilbert space \( H := H^2_E(0, 1) \times L^2(0, 1) \), where
\[
H^2_E(0, 1) := \{ f \in H^2(0, 1) : f(0) = f(1) = 0 \},
\]
endowed with an inner product norm in \( H \)
\[
\| (f, g) \|^2_H := \int_0^1 \left[ EI(x) |f''(x)|^2 + m(x) |g(x)|^2 \right] \, dx, \quad \forall (f, g) \in H,
\]
then (1.6) can be rewritten as an evolution equation in \( H \),
\[
\begin{aligned}
\frac{d}{dt} Y(t) &= A_\gamma Y(t), \\
Y(0) &= Y_0,
\end{aligned}
\]
where
\[
Y(t) := (u(\cdot, t), u_t(\cdot, t)), \quad Y(0) := (u_0, u_1),
\]
and \( A_\gamma \) is defined by
\[
A_\gamma (f, g) := \left( g, -\frac{1}{m(x)} \left( (EI(x)f''(x))'' + \gamma(x)g \right) \right), \quad \forall (f, g) \in D(A_\gamma),
\]
\[
D(A_\gamma) := \{ (f, g) \in (H^4 \cap H^2_E) \times H^2_E : f''(0) = f''(1) = 0 \}.
\]
Here, it is clear that \( A_0 \) denotes the undamped case \( \gamma(x) \equiv 0 \) and that \( \Gamma_\gamma := A_\gamma - A_0 \) is a bounded linear operator on \( H \). Therefore the following result follows immediately from the theory of linear operator semigroups (see, Pazy, 1983, Theorem 1.1).

**Theorem 2.1** Let \( A_\gamma \) and \( A_0 \) be defined as before. Then \( A_0 \) is a skew-adjoint operator and generates a \( C_0 \)-group on \( H \), and hence \( A_\gamma \) generates a \( C_0 \)-group \( e^{A_\gamma t} \) on \( H \).

**Theorem 2.2** \( A_\gamma \) has compact resolvents and \( 0 \in \rho(A_\gamma) \). Therefore, the spectrum \( \sigma(A) \) consists entirely of isolated eigenvalues.

**Proof.** Clearly, we only need to prove that \( 0 \in \rho(A_\gamma) \) and \( A_\gamma^{-1} \) is compact on \( H \). For any \( G := (g_1, g_2) \in H \), we need to find a unique \( F := (f_1, f_2) \in D(A_\gamma) \) such that
\[
A_\gamma F = G,
\]
which is the same as
\[
\begin{aligned}
f_2 &= g_1, \quad g_1 \in H^2_E(0, 1), \\
-\frac{1}{m(x)} [(EI(x)f''_1(x))'' + \gamma(x)f_2] &= g_2, \quad g_2 \in L^2(0, 1), \\
f(0) &= f(1) = f''(0) = f''(1) = 0.
\end{aligned}
\]
Solving the ordinary differential equation, we get
\[ f_1(x) = -\int_0^1 K(x, s) \frac{K(s, r)[\gamma(r)g_1(r) + m(r)g_2(r)] dr}{EI(s)} ds \]
with
\[ K(x, s) := \begin{cases} x(s - 1), & \text{if } s > x, \\ s(x - 1), & \text{if } s \leq x. \end{cases} \]

Obviously, \((f_1, f_2) \in D(A_y)\). Therefore
\[ F = (f_1, f_2) = A_y^{-1}G \]
\[ = \left( -\int_0^1 K(x, s) \frac{K(s, r)[\gamma(r)g_1(r) + m(r)g_2(r)] dr}{EI(s)} ds \right) g_1 \]
and Sobolev’s embedding theorem implies that \(A_y^{-1}\) is a compact operator on \(H\). The second assertion follows immediately from the first one.

Now we are ready to study the eigenvalue problem of \(A_y\). Let \(\lambda \in \sigma(A_y)\) and \(\psi := (\phi, \psi)\) be an eigenfunction of \(A_y\) corresponding to \(\lambda\). Then we have \(\psi = \lambda \phi\) and \(\phi\) satisfies the following equation:
\[
\begin{cases}
\lambda^2 m(x)\phi(x) + (EI(x)\phi''(x))'' + \lambda \gamma(x)\phi(x) = 0, & 0 < x < 1, \\
\phi(0) = \phi(1) = \phi''(0) = \phi''(1) = 0.
\end{cases}
\]

(2.8)

Expanding (2.8) yields
\[
\begin{aligned}
\phi^{(4)}(x) + & \frac{2EI'(x)}{EI(x)} \phi'''(x) + \frac{EI''(x)}{EI(x)} \phi''(x) \\
+& \lambda \frac{\gamma(x)}{EI(x)} \phi(x) + \lambda^2 \frac{m(x)}{EI(x)} \phi(x) = 0, & 0 < x < 1,
\end{aligned}
\]

(2.9)

In order to simplify our computations, we introduce a spatial-scale transformation in \(x\):
\[
f(z) := \phi(x), \quad z := \frac{1}{h} \int_0^x \left( \frac{m(\xi)}{EI(\xi)} \right)^{1/4} d\xi,
\]

(2.10)

where
\[
h := \int_0^1 \left( \frac{m(\xi)}{EI(\xi)} \right)^{1/4} d\xi.
\]

(2.11)

Then, (2.9) together with its boundary conditions can be transformed into
\[
\begin{cases}
f^{(4)}(z) + a(z)f'''(z) + b(z)f''(z) \\
+ c(z)f'(z) + \lambda h^4 [\gamma(x)/m(x)]f(z) + \lambda^2 h^4 f(z) = 0, & 0 < z < 1,
\end{cases}
\]

(2.12)

\[
\begin{aligned}
f''(1)z_x^2(1) + f'(1)z_{xx}(1) = 0, \\
f''(0)z_x^2(0) + f'(1)z_{xx}(0) = 0, \\
f(0) = f(1) = 0,
\end{aligned}
\]
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with

\[
a(z) := 6 \frac{z_{xx}}{z_x^2} + \frac{2EI'(x)}{z_x EI(x)},
\]

(2.13)

\[
b(z) := \frac{3z_{xx}^2}{z_x^4} + \frac{(6z_{xx} EI'(x))}{(z_x^2 EI(x))} + \frac{(EI''(x))}{(z_x^3 EI(x))} + \frac{4z_{xxx}}{z_x^3},
\]

(2.14)

\[
c(z) := \frac{z_{xxxx}}{z_x^4} + \frac{(2z_{xxx} EI'(x))}{(z_x^4 EI(x))} + \frac{(z_{xx} EI''(x))}{(z_x^4 EI(x))},
\]

(2.15)

\[
z_x := \frac{1}{h^4} \left( \frac{m(x)}{EI(x)} \right)^{1/4}, \quad z_x^4 = \frac{1}{h^3} \frac{m(x)}{EI(x)}
\]

(2.16)

and

\[
z_{xx} = \frac{1}{4h} \left( \frac{m(x)}{EI(x)} \right)^{-3/4} \frac{d}{dx} \left( \frac{m(x)}{EI(x)} \right)^{1/4}.
\]

(2.17)

If we define

\[
d(z) := \gamma(x) \frac{m(x)}{m(x)},
\]

(2.18)

then the equation in (2.12) is

\[
f^{(4)}(z) + a(z) f''''(z) + b(z) f''(z) + c(z) f'(z) + \lambda h^4 d(z) f(z) + \lambda^2 h^4 f(z) = 0, \quad 0 < z < 1.
\]

(2.19)

This can be further simplified by applying another invertible transformation:

\[
g(z) := \exp \left( \frac{1}{4} \int_0^z a(\xi) d\xi \right) f(z), \quad z \in (0, 1),
\]

(2.20)

and we arrive at the following eigenvalue problem that is equivalent to the original one:

\[
\begin{cases}
g^{(4)}(z) + b_1(z) g''(z) + c_1(z) g'(z) + d_1(z) g(z) + \lambda h^4 d(z) g(z) + \lambda^2 h^4 g(z) = 0, & 0 < z < 1, \\
g''(1) + b_{11} g'(1) + b_{12} g(1) = 0, \\
g''(0) + b_{21} g'(0) + b_{22} g(0) = 0, \\
g(0) = g(1) = 0,
\end{cases}
\]

(2.21)
where

\[ b_1(z) := -\frac{3}{2}a'(z) - \frac{3}{8}a^2(z) + b(z), \quad (2.22) \]

\[ b_{11} := \frac{\omega z_{xx}(1) - (1/2)z_{xx}^2(1)a(1)}{z_{xx}^2(1)} = -\frac{1}{2}a(1) + \frac{\omega z_{xx}(1)}{z_{xx}^2(1)}, \quad (2.23) \]

\[ b_{12} := \frac{1}{16}z_{xx}^2(1)a^2(1) - \frac{1}{4}z_{xx}^2(1)a'(1) - \frac{1}{4} \omega z_{xx}(1)a(1), \quad (2.24) \]

\[ b_{21} := \frac{\omega z_{xx}(0) - \frac{1}{2}z_{xx}^2(0)a(0)}{z_{xx}^2(0)} = -\frac{1}{2}a(0) + \frac{\omega z_{xx}(0)}{z_{xx}^2(0)}, \quad (2.25) \]

\[ b_{22} := \frac{1}{16}z_{xx}^2(0)a^2(0) - \frac{1}{4}z_{xx}^2(0)a'(0) - \frac{1}{4} \omega z_{xx}(0)a(0), \quad (2.26) \]

with abbreviations

\[ c_1(z) := c_1(a(z), b(z), c(z)), \quad d_1(z) := d_1(a(z), b(z), c(z)). \quad (2.27) \]

Here \( c_1(z) \) and \( d_1(z) \) are smooth functions of \( a(z), b(z) \) and \( c(z) \).

**Remark** The two terms \( c_1(z) \) and \( d_1(z) \) in (2.21) do not have any effect on computation of the high frequencies and hence we ignore their explicit expressions for simplicity.

To further solve the eigenvalue problem (2.21), we follow the procedure in Birkhoff (1908a,b) and Naimark (1967) and divide the complex plane into eight distinct sectors,

\[ S_k := \left\{ z \in \mathbb{C} : \frac{k\pi}{4} \leq \arg z \leq \frac{(k+1)\pi}{4} \right\}, \quad k = 0, 1, 2, \ldots, 7 \quad (2.28) \]

and let \( \omega_1, \omega_2, \omega_3, \omega_4 \) be the roots of equation \( \theta^4 + 1 = 0 \) that are arranged so that

\[ \text{Re}(\rho \omega_1) \leq \text{Re}(\rho \omega_2) \leq \text{Re}(\rho \omega_3) \leq \text{Re}(\rho \omega_4), \quad \forall \rho \in S_k. \quad (2.29) \]

Obviously, in sector \( S_1 \), we can choose

\[ \omega_1 = \exp(i\frac{\pi}{4}) = -\frac{\sqrt{2}}{2} - \frac{i\sqrt{2}}{2}, \quad \omega_2 = \exp(i\frac{3\pi}{4}) = \frac{\sqrt{2}}{2} + \frac{i\sqrt{2}}{2}, \quad (2.30) \]

which satisfy the inequalities in (2.29) and choices can also be made for other sectors. In the rest of this section, we shall derive the asymptotic behaviour of the eigenvalues in the sectors \( S_1 \) and \( S_2 \) because the same will hold for the other sectors with similar proofs.

Setting \( \lambda := \rho^2/h^2 \), in each sector \( S_k \), we have the following result about the asymptotic fundamental solutions of system (2.21).

**Lemma 2.1** For \( \rho \in S_k \) with \(|\rho| \) large enough, the equation

\[ g^{(4)}(z) + b_1(z)g''(z) + c_1(z)g'(z) + d_1(z)g(z) + \rho^2 \rho^2 d(z)g(z) + \rho^4 g(z) = 0, \quad (2.31) \]
has four linearly independent asymptotic fundamental solutions,

\[ y_i(z, \rho) := e^{\rho \omega_i z} \left( 1 + \frac{y_1(z)}{\rho} + \mathcal{O}(\rho^{-2}) \right), \quad i = 1, 2, 3, 4 \]  
(2.32)

and hence their derivatives for \( i = 1, 2, 3, 4 \) and \( j = 1, 2, 3 \) are given by

\[ \frac{d^j}{dz^j} y_i(z, \rho) = (\rho \omega_i)^j e^{\rho \omega_i z} \left( 1 + \frac{y_1(z)}{\rho} + \mathcal{O}(\rho^{-2}) \right), \]  
(2.33)

where

\[ y_1(z) = -\frac{1}{4\omega_i^3} \int_0^z (\omega_i^2 b_1(\xi) + h^2 d(\xi)) d\xi = -\frac{1}{4\omega_i} \int_0^z b_1(\xi) d\xi - \frac{h^2}{4\omega_i^3} \int_0^z d(\xi) d\xi. \]  
(2.34)

So for \( i = 1, 2, 3, 4 \),

\[ y_1(0) = 0 \]  
(2.35)

and

\[ y_1(1) = -\frac{1}{4\omega_i} \int_0^1 b_1(\xi) d\xi = -\frac{h^2}{4\omega_i^3} \int_0^1 d(\xi) d\xi = \frac{1}{\omega_i} \mu_1 + \frac{1}{\omega_i^2} \mu_2 = \frac{\omega_i^2 \mu_1 + \mu_2}{\omega_i^3} \]  
(2.36)

with

\[ \mu_1 := -\frac{1}{4} \int_0^1 b_1(\xi) d\xi \quad \text{and} \quad \mu_2 := -\frac{h^2}{4} \int_0^1 d(\xi) d\xi. \]  
(2.37)

\textbf{Proof.} The proof is a direct result in Birkhoff (1908a,b) and Naimark (1967). Here we briefly present a simple calculation to find the asymptotic expansions of fundamental solutions in sector \( S_k \). Let

\[ \tilde{y}_i(z, \rho) := e^{\rho \omega_i z} \left[ y_0(z) + \frac{y_1(z)}{\rho} \right], \quad i = 1, 2, 3, 4 \]

and

\[ D(g) := g^{(4)}(z) + b_1(z) g''(z) + c_1(z) g'(z) + d_1(z) g(z) + \rho^2 h^2 d(z) g(z) + \rho^4 g(z). \]
Then, by substituting $\tilde{y}_i(z, \rho)$ into the expression of $e^{-\rho \omega z} D(g)$ yields for $i = 1, 2, 3, 4$,

$$e^{-\rho \omega z} D(\tilde{y}_i(z, \rho)) = (\rho \omega)^4 \left[ y_{i0}(z) + \frac{y_{i1}(z)}{\rho} \right] + 4(\rho \omega)^3 \left[ y'_{i0}(z) + \frac{y'_{i1}(z)}{\rho} \right] + 6(\rho \omega)^2 \left[ y''_{i0}(z) + \frac{y''_{i1}(z)}{\rho} \right] + 4(\rho \omega) \left[ y'''_{i0}(z) + \frac{y'''_{i1}(z)}{\rho} \right] + b_1(z)(\rho \omega)^2 \left[ y_{i0}(z) + \frac{y_{i1}(z)}{\rho} \right] + b_2(z)(\rho \omega)^3 \left[ y'_{i0}(z) + \frac{y'_{i1}(z)}{\rho} \right] + b_3(z)(\rho \omega)^4 \left[ y''_{i0}(z) + \frac{y''_{i1}(z)}{\rho} \right] + (d_1(z) + \rho^2 \delta^2 d(z) + \rho^4) \left[ y_{i0}(z) + \frac{y_{i1}(z)}{\rho} \right]
$$

where $F_i(z, \rho)$ denote the remaining terms in the above equation and satisfy the estimates for some positive constant $M$:

$$|F_i(z, \rho)| \leq M, \quad x \in [0, 1].$$

Thus, by setting the coefficients of $\rho^3$ and $\rho^2$ to zero respectively, we obtain $y'_{i0}(z) = 0$ and

$$4\omega_i^3 y''_{i1}(z) + 6\omega_i^2 y''_{i0}(z) + b_1(z)\omega_i y_{i0}(z) + h^2 d(z) y_{i0}(z) = 0,$$

which yield that $y_{i0}(z) = 1$ and $y_{i1}(z)$ given in (2.34) are linearly independent solutions. Thus, as in the theorem in Birkhoff (1908a, pp. 225–226), we obtain the linearly independent fundamental solutions of (2.31) given by ($i = 1, 2, 3, 4$)

$$y_i(z, \rho) = \tilde{y}_i(z, \rho) + e^{\rho \omega z} \mathcal{O}(\rho^{-2}),$$

from which we deduce the required results (2.32) and (2.33).

For convenience, we introduce the notation

$$[a]_2 := a + \mathcal{O}(\rho^{-2}).$$

**Lemma 2.2** For $\rho \in \mathcal{S}_1$, if we set $\delta := \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$, then we have the following inequalities:

$$\Re(\rho \omega_i) \leq -|\rho| \delta, \quad \Re(\rho \omega_4) \geq |\rho| \delta.$$

Furthermore, substituting (2.32) and (2.33) into the boundary conditions (2.21), we obtain asymptotic expressions for the boundary conditions for large enough $|\rho|$:

$$U_4(y_i, \rho) = y_i(0, \rho) = 1 + \mathcal{O}(\rho^{-2}) := [1]_2, \quad i = 1, 2, 3,$$

(2.38)
determinant so substituting (2.38)–(2.42) into (2.43), we get

\[ \Delta(y, \rho) = y_i''(0, \rho) + b_{21}y_i'(0, \rho) + b_{22}y_i(0, \rho) \quad i = 1, 2, 3, \]

\[ = (\rho \omega_1)^2 \left( 1 + b_{21} \omega_1^{-1} \rho^{-1} + \mathcal{O}(\rho^{-2}) \right) \]
\[ := (\rho \omega_1)^2 \left[ 1 + b_{21} \omega_1^{-1} \rho^{-1} \right]_2, \quad (2.39) \]

\[ U_2(y, \rho) = y_1(1, \rho) = \exp(\rho \omega_1) \left[ 1 + (\omega_1^2 \mu_1 + \mu_2) \omega_1^{-3} \rho^{-1} + \mathcal{O}(\rho^{-2}) \right] \]
\[ := \exp(\rho \omega_1) \left[ 1 + (\omega_1^2 \mu_1 + \mu_2) \omega_1^{-3} \rho^{-1} \right]_2, \quad i = 2, 3, 4, \quad (2.40) \]

\[ U_1(y, \rho) = y_i''(1, \rho) + b_{11} y_i'(1, \rho) + b_{12} y_i(1, \rho) \]
\[ = (\rho \omega_1)^2 \exp(\rho \omega_1) \left[ 1 + (\omega_1^2 \mu_1 + \mu_2) \omega_1^{-3} + b_{11} \omega_1^{-1} \right] \rho^{-1} + \mathcal{O}(\rho^{-2}) \]
\[ := (\rho \omega_1)^2 \exp(\rho \omega_1) \left[ 1 + (\omega_1^2 \mu_1 + b_{11}) + \mu_2 \omega_1^{-3} \rho^{-1} \right]_2, \quad i = 2, 3, 4, \quad (2.41) \]

and

\[ U_4(y_4, \rho) = U_3(y_4, \rho) = U_2(y_1, \rho) = U_1(y_1, \rho) = 0. \quad (2.42) \]

**Proof.** The proof is a direct substitution. \( \square \)

Note that \( \lambda = \rho^2 / h^2 \neq 0 \) is the eigenvalue (2.21) if and only if \( \rho \) satisfies the characteristic determinant

\[ \Delta(\rho) = \begin{vmatrix} U_4(y_1, \rho) & U_4(y_2, \rho) & U_4(y_3, \rho) & U_4(y_4, \rho) \\ U_3(y_1, \rho) & U_3(y_2, \rho) & U_3(y_3, \rho) & U_3(y_4, \rho) \\ U_2(y_1, \rho) & U_2(y_2, \rho) & U_2(y_3, \rho) & U_2(y_4, \rho) \\ U_1(y_1, \rho) & U_1(y_2, \rho) & U_1(y_3, \rho) & U_1(y_4, \rho) \end{vmatrix} = 0, \quad (2.43) \]

so substituting (2.38)–(2.42) into (2.43), we get

\[ \Delta(\rho) = \begin{vmatrix} 1 & \rho \omega_1 \omega_1^{-1} \rho^{-1} \\ \rho \omega_1 \omega_1^{-1} \rho^{-1} & 1 \\ 0 & 0 \\ 0 & 0 \end{vmatrix} \]

\[ = \begin{vmatrix} 1 & \rho \omega_1 \omega_1^{-1} \rho^{-1} \\ \rho \omega_1 \omega_1^{-1} \rho^{-1} & 1 \\ \exp(\rho \omega_1) \left[ 1 + (\omega_1^2 \mu_1 + \mu_2) \omega_1^{-3} \rho^{-1} \right]_2 \\ \exp(\rho \omega_1) \left[ 1 + (\omega_1^2 \mu_1 + \mu_2) \omega_1^{-3} \rho^{-1} \right]_2 \end{vmatrix} \]

\[ = \begin{vmatrix} 1 & \rho \omega_1 \omega_1^{-1} \rho^{-1} \\ \rho \omega_1 \omega_1^{-1} \rho^{-1} & 1 \\ \exp(\rho \omega_1) \left[ 1 + (\omega_1^2 \mu_1 + \mu_2) \omega_1^{-3} \rho^{-1} \right]_2 \\ \exp(\rho \omega_1) \left[ 1 + (\omega_1^2 \mu_1 + \mu_2) \omega_1^{-3} \rho^{-1} \right]_2 \end{vmatrix} \]
In sector \( S_1 \), the choices are \( \omega_1^2 = -i, \omega_2^2 = i, \omega_3^2 = i \) and \( \omega_4^2 = -i \), so substituting them into (2.44), we have

\[
\Delta(\rho) = \rho^4 \exp(\rho \omega_4) \left\{ \left( -2i \right)^2 + \left( -2i \right) \times \left( b_{21}(\omega_1 - \omega_3) - (i\mu_1 + \mu_2)\omega_2^{-1} \right) + \left( -i(\mu_1 + b_{11}) + \mu_2 \right)\omega_4^{-1} + \left( -i(\mu_1 + b_{11}) + \mu_2 \right)\omega_3^{-1} + \left( -i(\mu_1 + b_{11}) + \mu_2 \right)\omega_3^{-1} \right\} \exp(\rho \omega_2) + \mathcal{O}(\rho^{-2}) \}
\]
and hence
\[ \Delta(\rho) = \rho^4 \exp(\rho \omega_4) \left\{ -4 + \left( 2ib_{21}(\omega_3 - \omega_1) + (4i\mu_1 + 2ib_{11})(\omega_2 - \omega_4) \right) \\
+ 4\mu_2(\omega_2 + \omega_4) \rho^{-1} \right\} \exp(\rho \omega_2) + \left[ 4 - \left( 2ib_{21}(\omega_2 - \omega_1) \right) \\
+ (4i\mu_1 + 2ib_{11})(\omega_3 - \omega_4) + 4\mu_2(\omega_3 + \omega_4) \right] \rho^{-1} \exp(-\rho \omega_2) + O(\rho^{-2}) \right\}. \tag{2.45} \]

Since
\[ \omega_2 - \omega_1 = \sqrt{2}, \quad \omega_3 - \omega_4 = -\sqrt{2}, \]
\[ \omega_3 + \omega_4 = -\sqrt{2}i, \quad \omega_1 - \omega_2 = -\sqrt{2}i, \]
\[ \omega_2 - \omega_3 = \sqrt{2i}, \quad \omega_2 + \omega_3 = \sqrt{2}, \]
a straightforward simplification will arrive at the following result, which is also true on all the other sectors \( S_k \) (see Naimark, 1967, pp. 56–74).

**Theorem 2.3** Let \( \Delta(\rho) \) be the characteristic determinant of the eigenvalue problem (2.21). In sector \( S_1 \), an asymptotic expression of \( \Delta(\rho) \) is given by
\[ \Delta(\rho) = \rho^4 e^{\rho \omega_4} \left\{ -4e^{\rho \omega_2} + 4e^{-\rho \omega_2} + 4\mu_3 \rho^{-1} e^{\rho \omega_2} + 4i\mu_4 \rho^{-1} e^{-\rho \omega_2} + O(\rho^{-2}) \right\}, \tag{2.46} \]

where
\[ \mu_3 := (\sqrt{2}/2)(b_{21} - b_{11}) + \sqrt{2}(\mu_2 - \mu_1), \]
\[ \mu_4 := (\sqrt{2}/2)(b_{11} - b_{21}) + \sqrt{2}(\mu_1 + \mu_2). \tag{2.47} \]

Thus, the boundary eigenvalue problem (2.21) is strongly regular.

Using (2.46), we can deduce an asymptotic expression for the eigenvalues of problem (2.21). The equation \( \Delta(\rho) = 0 \) and (2.46) imply that
\[ -4e^{\rho \omega_2} + 4e^{-\rho \omega_2} + 4\mu_3 \rho^{-1} e^{\rho \omega_2} + 4i\mu_4 \rho^{-1} e^{-\rho \omega_2} + O(\rho^{-2}) = 0 \]
which is equivalent to
\[ e^{\rho \omega_2} - e^{-\rho \omega_2} - \mu_3 \rho^{-1} e^{\rho \omega_2} - i\mu_4 \rho^{-1} e^{-\rho \omega_2} + O(\rho^{-2}) = 0 \tag{2.48} \]
and can be rewritten as
\[ e^{\rho \omega_2} - e^{-\rho \omega_2} + O(\rho^{-1}) = 0. \tag{2.49} \]

Note that the following equation:
\[ e^{\rho \omega_2} - e^{-\rho \omega_2} = 0 \]
has solutions
\[ \rho_k = \frac{1}{\omega_2} k \pi i, \quad k = 1, 2, \ldots. \tag{2.50} \]
Let \( \tilde{\rho}_k \) be the solutions of (2.49). Applying Rouché’s theorem (see Naimark, 1967, p. 70) to (2.49), we get the following expression:

\[
\tilde{\rho}_k = \rho_k + \alpha_k = \frac{1}{\omega_2} k\pi i + \alpha_k, \quad \alpha_k = \mathcal{O}(k^{-1}), \quad k = N, N + 1, \ldots. \tag{2.51}
\]

where \( N \) is a large positive integer. Substituting \( \tilde{\rho}_k \) into (2.48), and using the fact that \( \exp(\rho_k \omega_2) = \exp(-\rho_k \omega_2) \), we obtain

\[
e^{\alpha_1 \omega_2} - e^{-\alpha_1 \omega_2} - \mu_3 \tilde{\rho}_k^{-1} e^{\alpha_1 \omega_2} - i\mu_4 \tilde{\rho}_k^{-1} e^{-\alpha_1 \omega_2} + \mathcal{O}(\tilde{\rho}_k^{-2}) = 0.
\]

Expanding the exponential function according to its Taylor series, we get

\[
\alpha_k = \frac{\mu_3}{2\rho_k \omega_2} + \frac{i\mu_4}{2\rho_k \omega_2} + \mathcal{O}(k^{-2}).
\]

Therefore, we have

\[
\tilde{\rho}_k = \frac{1}{\omega_2} k\pi i + \frac{\mu_3}{2} \frac{1}{k\pi} + \frac{i\mu_4}{2} \frac{1}{k\pi} + \mathcal{O}(k^{-2}), \quad k = N, N + 1, \ldots.
\]

Note that \( \lambda_k = \rho_k^2 / h^2, \omega_2 = \exp(i\frac{\pi}{2}) \) and \( \omega_2^2 = i \). So we have

\[
\lambda_k = \frac{1}{h^2} \frac{\sqrt{2}}{2} (\mu_3 + \mu_4) + \frac{1}{h^2} \left[ \frac{\sqrt{2}}{2} (\mu_4 - \mu_3) + (k\pi)^2 \right] i + \mathcal{O}(k^{-1}), \quad k = N, N + 1, \ldots. \tag{2.52}
\]

with \( N \) large enough.

The same proof can be applied to sector \( S_2 \) because the eigenvalues of the problem (2.21) can be obtained by a similar calculation with the choices

\[
\omega_1 = \exp(i\frac{\pi}{2}) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i, \quad \omega_2 = \exp(i\frac{3\pi}{2}) = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i, \\
\omega_3 = \exp(i\frac{5\pi}{2}) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i, \quad \omega_4 = \exp(i\frac{7\pi}{2}) = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i
\]

so that inequality (2.29) holds:

\[
\text{Re}(\rho_1 \omega_1) \leq \text{Re}(\rho_2 \omega_2) \leq \text{Re}(\rho_3 \omega_3) \leq \text{Re}(\rho_4 \omega_4), \quad \forall \rho \in S_2.
\]

Hence, in sector \( S_2 \), we have the following asymptotic expression of \( \Delta(\rho) \):

\[
\Delta(\rho) = \rho^4 e^{\rho \omega_4} \left[ -4e^{\rho \omega_2} + 4e^{-\rho \omega_2} - 4\mu_3 \rho^{-1} e^{\rho \omega_2} + 4i\mu_4 \rho^{-1} e^{-\rho \omega_2} + \mathcal{O}(\rho^{-2}) \right]. \tag{2.54}
\]

By a direct calculation, we have

\[
\tilde{\rho}_k = -\frac{1}{\omega_2} k\pi i + \frac{\mu_3}{2} \frac{1}{k\pi} + \frac{i\mu_4}{2} \frac{1}{k\pi} + \mathcal{O}(k^{-2}), \quad k = N, N + 1, \ldots,
\]

with \( N \) large enough. Again, using \( \lambda_k = \rho_k^2 / h^2, \omega_2 = \exp(i\frac{3\pi}{2}) \) and \( \omega_2^2 = -i \), we obtain the conjugate eigenvalues of the problem (2.21), namely

\[
\lambda_k = \frac{1}{h^2} \frac{\sqrt{2}}{2} (\mu_3 + \mu_4) - \frac{1}{h^2} \left[ \frac{\sqrt{2}}{2} (\mu_4 - \mu_3) + (k\pi)^2 \right] i + \mathcal{O}(k^{-1}), \quad k = N, N + 1, \ldots. \tag{2.55}
\]
From (2.47), we have
\[ \frac{\sqrt{2}}{2} (\mu_3 + \mu_4) = 2\mu_2, \quad \text{and} \quad \frac{\sqrt{2}}{2} (\mu_4 - \mu_3) = (b_{11} - b_{21}) + 2\mu_1. \] (2.56)

Here we should point out that the eigenvalues generated from the other sectors \( S_k \) coincide with those from \( S_1 \) and \( S_2 \) (the detailed argument can be found in Naimark, 1967, pp. 56–74). Combining with (2.52) and (2.55), we obtain the following result on the eigenvalues.

**Theorem 2.4** Let \( A_\gamma \) be defined by (2.5) and (2.6), then an asymptotic expression of the eigenvalues of the problem (2.21) is given explicitly by

\[ \lambda_k = \frac{2}{h^2} \mu_2 \pm \frac{1}{h^2} \left[ (b_{11} - b_{21}) + 2\mu_1 + (k\pi)^2 \right] i + \mathcal{O}(k^{-1}), \quad k = N, N + 1, \ldots, \] (2.57)

where \( N \) is a large positive integer, and

\[ \mu_2 = -\frac{h^2}{4} \int_0^1 d(\zeta) d\xi, \quad \text{d}(z) = \frac{\gamma(x)}{m(x)}, \quad \frac{dz}{dx} = \frac{1}{h} \left( \frac{m(x)}{EI(x)} \right)^{1/4}, \]

so,

\[ \mu_2 = -\frac{h^2}{4} \int_0^1 \frac{\gamma(x)}{m(x)} \left( \frac{m(x)}{EI(x)} \right)^{1/4} dx = -\frac{h}{4} \int_0^1 \frac{\gamma(x)}{m(x)} \left( \frac{m(x)}{EI(x)} \right)^{1/4} dx. \] (2.58)

Moreover, \( \lambda_k \) \((k = N, N + 1, \ldots)\) with sufficiently large modulus are simple and distinct except for finitely many of them, and satisfy

\[ \text{Re} \lambda_k \to -\frac{1}{2} \int_0^1 \frac{\gamma(x)}{m(x)} \left( \frac{m(x)}{EI(x)} \right)^{1/4} dx, \quad \text{as} \quad k \to \infty. \] (2.59)

**Remark** In Theorem 2.4, we obtain the asymptotic expression of the eigenvalues of \( A_\gamma \). If \( \lambda \in \sigma(A_\gamma) \) with \( |\lambda| \) large enough, then \( \lambda \) has an asymptotic form given by (2.57). So, \( \sigma(A_\gamma) \setminus \{ \lambda_k, \bar{\lambda}_k \mid k \geq N \} \) has only finitely many eigenvalues.

**3. Riesz basis property of the eigenfunctions of \( A_\gamma \) and exponential stability**

In this section, we discuss the Riesz basis property of the eigenfunctions of \( A_\gamma \) and the exponential stability of system (2.3). We begin with showing that the generalized eigenfunctions of \( A_\gamma \) form an unconditional basis in Hilbert state space \( \mathcal{H} \).

For this aim, we introduce a transformation \( T \) via

\[ T(f, g) = (\phi, \psi) \]

where

\[ \phi(x) = f(z), \quad \psi(x) = g(z), \quad z = \frac{1}{h} \int_0^x \left( \frac{m(\zeta)}{EI(\zeta)} \right)^{1/4} d\zeta, \quad (3.1) \]
with

\[
h = \int_0^1 \left( \frac{m(\xi)}{EI(\xi)} \right)^{1/4} \, d\xi.
\]  

(3.2)

It is easily seen that \( T \) is a bounded invertible operator on \( \mathcal{H} \).

Now we define the following ordinary differential operator:

\[
\begin{align*}
L(f) &= f^{(4)}(z) + a(z)f''(z) + b(z)f'(z) + c(z)f(z), \\
\mu(z) &= h^2d(z) = h^2 \frac{\gamma(x)}{m(x)}, \\
B_1(f) &= f(0), \quad B_2(f) = f(1), \\
B_3(f) &= f''(1)\zeta_2^2(1) + f'(1)\zeta_{xx}(1) = 0, \\
B_4(f) &= f''(0)\zeta_2^2(0) + f'(1)\zeta_{xx}(0) = 0,
\end{align*}
\]

(3.3)

where the coefficients are given by (2.13)–(2.17). Let \( A \) be defined as in (1.5), \( \xi \in \sigma(A) \) be an eigenvalue of \( A \) and \((f, g)\) be an eigenfunction, then we have \( g = \xi f \) and \( f \) will satisfy the following equation:

\[
f^{(4)}(z) + a(z)f''(z) + b(z)f'(z) + c(z)f(z) + \xi \mu(z)f(z) + \xi^2 f(z) = 0,
\]

with boundary conditions \( B_j(f) = 0, \ j = 1, 2, 3, 4 \). Now by taking \( \lambda = \xi / h^2 \) and

\[
T(f, g) = (\phi(x), \psi(x)),
\]

we see that \( \psi = \lambda \phi \) and \( \phi \) satisfies the equation

\[
\begin{align*}
\phi^{(4)}(x) + \frac{2EI'(x)}{EI(x)} \phi''(x) + \frac{EI''(x)}{EI(x)} \phi''(x) + \lambda \frac{\gamma(x)}{EI(x)} \phi(x) + \lambda^2 \frac{m(x)}{EI(x)} \phi(x) &= 0, \quad 0 < x < 1, \\
\phi(0) &= \phi(1) = \phi''(0) = \phi''(1) = 0.
\end{align*}
\]

(3.4)

Hence we have the following result.

**Theorem 3.1** Let \( A_\gamma \) be defined by (2.5) and (2.6). Then the eigenvalues of operator \( A_\gamma \) are all simple except for finitely many of them, and the generalized eigenfunctions of \( A_\gamma \) form a Riesz basis for the Hilbert state space \( \mathcal{H} \).

**Proof.** From the previous section, we have shown that the boundary problem (2.21) is strongly regular, which implies that the eigenvalues are separated and simple except for finitely many of them. Thus the first assertion is true. Next, according to Theorem 1.1, the strongly regular boundary conditions ensure that the generalized eigenfunction sequence \( F_n = (f_n, \xi_n f_n) \) of operator \( A \) forms a Riesz basis for \( \mathcal{H} \).

Since \( T \) is bounded and invertible on \( \mathcal{H} \), it follows that \( \psi_n = (\phi_n, \lambda_n \phi_n) = TF_n \) also forms a Riesz basis in \( \mathcal{H} \).

We are now in a position to investigate the stability of system (2.3). Since the Riesz basis property implies the spectrum-determined growth condition (see Curtain & Zwart, 1995), and (2.59) describes the asymptote of \( \sigma(A_\gamma) \), for any small \( \varepsilon > 0 \) there are only finitely many eigenvalues of \( A_\gamma \) in the following half-plane:

\[
\Sigma : \quad \text{Re} \lambda > -\frac{1}{2h} \int_0^1 \frac{\gamma(x)}{m(x)} \left( \frac{m(x)}{EI(x)} \right)^{1/4} \, dx + \varepsilon.
\]

(3.5)
The following are two stability results that describe how stability depends upon the damping function $\gamma$.

**Theorem 3.2** If $\gamma(x) \geq 0$ is continuous and there exists an open set $I \subset (0, 1)$ such that $\gamma|_I > 0$, then the system (2.3) is exponentially stable.

**Proof.** We have $\gamma(x) \geq 0$, and for any $(f, g) \in D(A_\gamma)$,

$$\langle A_\gamma(f, g), (f, g) \rangle = \left\langle \left( g, -\frac{1}{m(x)} \left( (EI(x)f''(x))'' + \gamma(x)g \right) \right), (f, g) \right\rangle$$

$$= \int_0^1 \left[ EI(x)g''(x)f''(x) - (EI(x)f''(x))'' \frac{g(x)}{g(x)} - \gamma(x)|g(x)|^2 \right] \, dx$$

$$= \int_0^1 EI(x) \left[ g''(x)f''(x) - f''(x)g''(x) \right] - \gamma(x)|g(x)|^2 \, dx,$$

further

$$\text{Re} \langle A_\gamma(f, g), (f, g) \rangle = -\int_0^1 \gamma(x)|g(x)|^2 \, dx \leq 0.$$

Thus $A_\gamma$ is dissipative and $e^{A_\gamma t}$ is a contraction semigroup on $H$. Moreover, the spectrum of $A_\gamma$ has an asymptote

$$\text{Re} \lambda \sim -\frac{1}{2h} \int_0^1 \gamma(x) \left( \frac{m(x)}{m(x)} \right)^{1/2} \, dx.$$

If we can show that there is no eigenvalue on the imaginary axis then the exponential stability holds. Let $\lambda = ir$ with $r \in \mathbb{R}$ be an eigenvalue of $A_\gamma$ on the imaginary axis and let $\Psi = (\phi, \psi)$ be the corresponding eigenfunction. Then we have

$$0 = \text{Re} \langle A_\gamma \Psi, \Psi \rangle = -\int_0^1 \gamma(x)|\psi(x)|^2 \, dx. \quad (3.6)$$

Note that $\gamma(x)$ and $\psi(x)$ are continuous functions, so

$$\gamma(x)|\psi(x)| = 0, \quad \forall x \in (0, 1).$$

Since $\gamma|_I > 0$, we have

$$\psi|_I \equiv 0,$$

which implies that the eigenfunction $\Psi = (\phi, \psi)$ of $A_\gamma$ is zero because $\psi = \lambda \phi$ and $\phi(x)$ has to obey the uniqueness theorem of the differential equation

$$\begin{cases}
\lambda^2 m(x) \phi(x) + (EI(x)\phi''(x))'' + \lambda \gamma(x) \phi(x) = 0, & 0 < x < 1, \\
\phi(0) = \phi(1) = \phi''(0) = \phi''(1) = 0, \\
\phi(x) = 0, & x \in I.
\end{cases} \quad (3.7)$$

However, $\psi = 0$ contradicts $\Psi$ being an eigenfunction and so there is no eigenvalue on the imaginary axis.

From Theorem 3.1 and the spectrum-determined growth condition, the system is exponentially stable. \qed
Now we are ready to consider the case that $\gamma(x)$ is continuous and indefinite in $[0, 1]$. Let

$$\gamma_+(x) := \max[\gamma(x), 0], \quad \gamma_-(x) := \max[-\gamma(x), 0]$$

and let

$$A_{\gamma_+}(f, g) := \left( g - \frac{1}{m(x)} \left( (EI(x)f''(x))'' + \gamma_+(x)g \right) \right), \quad \forall (f, g) \in D(A_{\gamma_+}) = D(A_{\gamma})$$

and

$$\Gamma_-(f, g) := \left( 0, \frac{\gamma_-(x)}{m(x)} g(x) \right), \quad \forall (f, g) \in H.$$ 

Then $A_{\gamma}$ can be written as $A_{\gamma} = A_{\gamma_+} + \Gamma_-.$

**Theorem 3.3** Let $s(A_{\gamma_+}) := \sup \{ \Re \lambda \mid \lambda \in \sigma(A_{\gamma_+}) \}$. If

$$\max_{x \in [0, 1]} \left\{ \frac{\gamma_-(x)}{m(x)} \right\} < |s(A_{\gamma_+})|,$$  \hspace{1cm} (3.8)  

then system (2.3) is exponentially stable.

**Proof.** It is easy to verify that $\Gamma_-$ is a self-adjoint operator and

$$\|\Gamma_\|= \max_{x \in [0, 1]} \left\{ \frac{\gamma_-(x)}{m(x)} \right\}.$$ 

By Theorem 3.2 and definition of operator $A_{\gamma_+}$, $e^{A_{\gamma_+} t}$ is a contraction semigroup and $s(A_{\gamma_+}) < 0$. Applying the perturbation theory of linear operators semigroup (see, Pazy, 1983, Theorem 1.1), we have $\lambda \in \rho(A_{\gamma})$ whenever $\Re \lambda > s(A_{\gamma_+}) + \|\Gamma_\|$.

Again, Theorem 3.1 gives

$$s(A_{\gamma}) = s(A_{\gamma_+}) + \|\Gamma_\| < 0,$$

where $\omega(A_{\gamma})$ denotes the growth bound of the semigroup $e^{A_{\gamma} t}$. Therefore, system (2.3) is exponentially stable. \hfill \Box

**Remark** Theorem 3.3 provides an answer via condition (3.8) to the question ‘How much more positive than negative should the damping be without disrupting the exponential stability?’ raised in papers such as Freitas & Zuazua (1996), Shubov (1999), Liu et al. (2001), Benaddi & Rao (2000). Although a complete answer to this question is far from achieved in this theorem, we believe condition (3.8) is a good start as well as a first attempt to answer this question. Furthermore, the spectral bound $s(A_{\gamma_+})$ can be computed either by examining the characteristic equation of $A_{\gamma_+}$ following our procedure in the first few sections, or by carrying out a numerical calculation such as those in Data et al. (2000, 2002), Liew et al. (1995), Mizusawa (1993) or Mutsuda & Sakiyama (1988).

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