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Exponential stability of variable coefficients Rayleigh beams under boundary feedback controls: a Riesz basis approach

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Abstract

In this paper, we study the boundary stabilizing feedback control problem of Rayleigh beams that have non-homogeneous spatial parameters. We show that no matter how non-homogeneous the Rayleigh beam is, as long as it has positive mass density, stiffness and mass moment of inertia, it can always be exponentially stabilized when the control parameters are properly chosen. The main steps are a detail asymptotic analysis of the spectrum of the system and the proving of that the generalized eigenfunctions of the feedback control system form a Riesz basis in the state Hilbert space. As a by-product, a conjecture in Guo (J. Optim. Theory Appl. 112(3) (2002) 529) is answered.

Keywords: Rayleigh beam; Eigenvalue distributions; Riesz basis; Exponential stability

1. Introduction

In this paper, we study non-homogeneous Rayleigh beam governed by

$$\begin{aligned}
&\rho(x)u_{tt} - (I_\rho(x)u_{ttx})_x + (EI(x)u_{xx})_{xx} = 0, \quad 0 < x < 1,\ t > 0,\\
&u(0,t) = u_x(0,t) = 0, \quad t > 0,\\
&EI(1)u_{xx}(1,t) + \alpha u_{xt}(1,t) = 0, \quad t > 0,\\
&(EIu_{xx})_x(1,t) - I_\rho(1)u_{xtt}(1,t) - \beta u_5(1,t) = 0, \quad t > 0,\\
&u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x).
\end{aligned} \tag{1.1}$$

The situations where the coefficients are non-constant arise in engineering problems that use non-homogeneous materials, such as smart materials. Here, $u(x,t)$ is the transverse displacement and x,t stand respectively for the position and time, and $\rho(x) > 0$ is the mass density, $EI(x) > 0$ is the stiffness of the beam, $I_\rho(x) > 0$ is the mass moment of inertia and $\alpha, \beta \geqslant 0$, which can be tuned, are constant feedback gains of the actuators

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with collocated measurements at the tip ends. A very nice description of the physical background as well as earlier development of (1.1) can be found in [9] and we would like to refer the interested readers there for other details of this model.

The constant coefficient version of (1.1) has been investigated in [8,2] say, where the former established the exponential stability under the condition $\alpha=1, \beta \geqslant 0$ and the later proved a Riesz basis property. In this paper, we shall extend these results to the variable coefficient cases with less restrictive conditions on the parameters. In fact, we show that no matter how non-homogeneous the Rayleigh beam is, as long as it has positive mass density, stiffness and mass moment of inertia, it can always be exponentially stabilized when the control parameters are properly chosen. Our investigation concentrates on proving that the generalized eigenfunctions of the feedback system form a Riesz basis in the state Hilbert space. Although there are many works in the literatures dealing with the Riesz basis of vibration systems, for example [3] for Euler-Bernoulli beam and [10,14] for Timoshenko beam, our method to obtain Riesz basis is different from that used in [2,3,10,14], among them, the authors mainly proved that the generalized eigenfunctions of the systems are quadratically close to a reference Riesz basis system, and then used Bari's Theorem to complete the proof of the Riesz basis generation. Here we treat the verification of Riesz basis property by the exponential family (see [15]). The advantage of this method lies on which it need not to compute the generalized eigenfunctions as long as the distribution and the separability of the eigenvalues of system (1.1) are known. As a consequence, we also provide an answer for the conjecture in [2].

It is well known that the study of variable coefficient differential equations with (control) parameters is difficult because explicit solution formula is hard to come by. One of the two crucial steps in this paper to overcome this obstacle is the proving of the Riesz basis property because then the system will satisfy the spectrum determined growth condition and stability can be deduced directly from an spectral analysis, which becomes the second crucial step because the classical treatment [7] alone is not working and it has to combine with the method of operator pencil in [1,11,12] to obtain asymptotic expansions for the fundamental solutions, the characteristic determinant and the eigenvalues of problem (1.1).

The rest of the paper is organized as follows. In Section 2 we convert system (1.1) into an evolution equation in an appropriate Hilbert space and then prove that the evolutionary system is associated to a C_0 semigroup of linear operators whose generator has compact resolvents. Hence, the problem is well-posed. In Section 3, we recall from [13] an asymptotic distribution of the eigenvalues and this forms our foundation to prove the Riesz basis property and the exponential stability of the system in the last section. Finally, a conjecture in [2] is discussed and answered.

Throughout this paper, we always assume that

$$\rho(x), I_\rho(x), EI(x) \in C^4[0,1]. \tag{1.2}$$

2. Hilbert space setup and eigenvalue problem

We start our investigation by formulating the problem in the following Hilbert spaces: $V := \{f \in H^1(0,1) \mid f(0)=0\}$ with norm $\|f\|_V^2 := \int_0^1 [\rho(x)|f(x)|^2 + I_\rho(x)|f'(x)|^2]\,\mathrm{d}x$ and $W := \{f \in H^2(0,1) \mid f(0)=f'(0)=0\}$ with norm $\|f\|_W^2 = \int_0^1 EI(x)|f''(x)|^2\,\mathrm{d}x$. Easy to see that

$$W \subset V \subset L^2[0,1] \subset V' \subset W',$$

where W' and V' are the dual spaces of W and V, respectively.

We now define linear operators $A, D \in \mathscr{L}(W, W')$ and $B, C \in \mathscr{L}(V, V')$ by

$$\langle A\phi, \psi\rangle = \int_0^1 EI(x)\phi''(x)\overline{\psi''(x)}\,\mathrm{d}x, \quad \forall \phi, \psi \in W, \tag{2.1}$$

$$\langle D\phi, \psi\rangle = \phi'(1)\overline{\psi'(1)}, \quad \forall \phi, \psi \in W, \tag{2.2}$$

$$\langle B\phi, \psi\rangle = \phi(1)\overline{\psi(1)}, \quad \forall \phi, \psi \in V, \tag{2.3}$$

$$\langle C\phi, \psi\rangle = \int_0^1 [\rho(x)\phi(x)\overline{\psi(x)} + I_\rho(x)\phi'(x)\overline{\psi'(x)}]\,dx, \quad \forall \phi, \psi \in V. \tag{2.4}$$

Lax–Milgram Theorem [16, p. 92] says that A (resp. C) is a canonical isomorphism of W (resp. V) onto W' (resp. V'). With these operators, Eq. (1.1) can be written into a variational equation

$$\langle Cu_{tt}, \psi\rangle + \langle Au, \psi\rangle + \alpha\langle Du_t, \psi\rangle + \beta\langle Bu_t, \psi\rangle = 0, \quad \forall \psi \in W. \tag{2.5}$$

Let $\mathscr{H} := W \times V$ with norm $\|(f,g)\|^2_{\mathscr{H}} := \|f\|^2_W + \|g\|^2_V$. Then we can define a linear operator $\mathscr{A}$ on $\mathscr{H}$ by

$$\mathscr{D}(\mathscr{A}) := \{(f,g) \in \mathscr{H} \mid g \in W, Af + \alpha Dg + \beta Bg \in V'\}, \tag{2.6}$$

$$\mathscr{A}(f,g) := (g, -C^{-1}(Af + \alpha Dg + \beta Bg)), \quad \forall (f,g) \in \mathscr{D}(\mathscr{A}). \tag{2.7}$$

Thus, (2.5) can be formulated into an evolution equation in $\mathscr{H}$ (with $Y(t) := (u, u_t)$)

$$\frac{dY(t)}{dt} = \mathscr{A}Y(t), \quad t > 0, \quad Y(0) = Y_0 = (u_0, u_1). \tag{2.8}$$

Lemma 2.1. *Let $\mathscr{A}$ be defined by* (2.6) *and* (2.7). *Then $\mathscr{A}$ is a densely defined closed dissipative operator with $0 \in \rho(\mathscr{A})$, and so $\mathscr{A}$ generates a C_0 semigroup of contraction.*

Proof. For any $(f,g) \in \mathscr{D}(\mathscr{A})$, we have

$$\begin{aligned}\langle \mathscr{A}(f,g), (f,g)\rangle &= \int_0^1 EI(x)g''(x)\overline{f''(x)}\,dx - \int_0^1 [Af + \alpha Dg + \beta Bg]\overline{g(x)}\,dx \\ &= \langle Ag, f\rangle - \langle Af, g\rangle - \alpha\langle Dg, g\rangle - \beta\langle Bg, g\rangle \\ &= \langle Ag, f\rangle - \langle Af, g\rangle - \alpha|g'(1)|^2 - \beta|g(1)|^2.\end{aligned}$$

So

$$\mathrm{Re}\langle \mathscr{A}(f,g), (f,g)\rangle = -\alpha|g'(1)|^2 - \beta|g(1)|^2 \leqslant 0.$$

To show that $0 \in \rho(\mathscr{A})$, we let $(y,z) \in \mathscr{H}$ and consider the resolvent equation

$$\mathscr{A}(f,g) = (y,z),$$

which is equivalent to $y = g$ and $z = -C^{-1}[Af + \alpha Dg + \beta Bg]$. So, for any $\psi \in W$,

$$\langle Af + \alpha Dg + \beta Bg, \psi\rangle = -\langle Cz, \psi\rangle.$$

Substituting $g = y$ back into the above equation yields

$$\langle Af, \psi\rangle = -\langle Cz + \alpha Dy + \beta By, \psi\rangle, \quad \forall \psi \in W. \tag{2.9}$$

Since for any $\psi \in W$, we have $\langle A\psi, \psi\rangle = \|\psi\|_W^2$. So from the Lax–Milgram Theorem, there exists a unique $f \in W$ so that (2.9) holds and $0 \in \rho(\mathscr{A})$ by letting $g := y$. The remaining part of the lemma is a direct consequence from the theory of semigroup of operators (see, [4, p. 3, Theorem 1.2.4]). □

Lemma 2.2. *Let $(f,g) \in \mathscr{H}$. Then $(f,g) \in \mathscr{D}(\mathscr{A})$ if and only if $f \in W \cap H^3$ and $g \in W$ such that*

$$(EI(x)f''(x))'|_{x=1} + I_\rho[C^{-1}(Af, \alpha Dg + \beta Bg)]'|_{x=1} - \beta g(1) = 0,$$
$$EI(1)f''(1) + \alpha g'(1) = 0.$$

From this we see that $\mathscr{A}^{-1}$ is compact.

Proof. The sufficiency is obvious. To prove the necessity, let $(f,g) \in \mathscr{D}(\mathscr{A})$ and $\mathscr{A}(f,g) = (y,z) \in \mathscr{H}$. Then we have $g = y \in W$ and

$$-C^{-1}[Af + \alpha Dg + \beta Bg] = z.$$

Since $z \in V$ and $C : V \to V'$ is an isomorphism, so we have

$$Af + \alpha Dy + \beta By = -Cz, \quad \text{in } V' \subset W'$$

and hence

$$\int_0^1 EIf''\bar{\psi}''\,dx + \alpha y'(1)\overline{\psi'(1)} + \beta y(1)\overline{\psi(1)} + \int_0^1 [\rho z\bar{\psi} + I_\rho z'\bar{\psi}']\,dx = 0, \quad \forall \psi \in W. \tag{2.10}$$

Now for any $\phi \in C_0^\infty(0,1)$, let $\psi(x) = \int_0^x \phi(s)\,ds$ and substitute it into (2.10) yields

$$\int_0^1 EIf''\bar{\phi}'\,dx + \beta y(1)\int_0^1 \bar{\phi}\,dx + \int_0^1 \bar{\phi}\,dx \int_x^1 \rho z\,ds + \int_0^1 I_\rho z'\bar{\phi}\,dx = 0$$

and

$$\int_0^1 EIf''\bar{\phi}'\,dx = -\int_0^1 \left[\beta y(1) + \int_x^1 \rho z\,ds + I_\rho z'\right]\bar{\phi}\,dx,$$

for all $\phi \in W$. Thus

$$(EIf'')' = \beta y(1) + \int_x^1 \rho z\,ds + I_\rho z' \in L^2[0,1]. \tag{2.11}$$

Since $EI \in C^4[0,1]$ (see (1.2)), so we have $f \in H^3(0,1) \cap W$. In particular, $(EIf'')'|_{x=1} = \beta y(1) + I_\rho z'|_{x=1}$. Inserting $g = y$ and $z = -C^{-1}(Af + \alpha Dg + \beta Bg)$ into the above yields

$$(EIf'')'|_{x=1} + I_\rho[C^{-1}(Af + \alpha Dg + \beta Bg)]'|_{x=1} - \beta g(1) = 0.$$

Again, for $\phi \in V$ with $\phi(1) = 1$, we let $\psi := \int_0^x \phi(s)\,ds$, and plug it into (2.10) and conclude from (2.11) that $EI(1)f''(1) + \alpha y'(1) = 0$. The necessity is also proved because $g = y$. By Lemma 2.1, $\mathscr{A}^{-1}$ exists and is bounded on $\mathscr{H}$. From the Sobolev Embedding Theorem, $\mathscr{A}^{-1}$ is compact. □

We are now in a position to recall some results for the eigenvalues of $\mathscr{A}$ that have been announced in [13]. Let $\lambda \in \sigma(\mathscr{A})$ and $(\phi, \psi) \in \mathscr{H}$ be such that $\mathscr{A}(\phi, \psi) = \lambda(\phi, \psi)$. Then, $\psi = \lambda\phi$ and ϕ satisfies

$$\lambda^2 \rho(x)\phi(x) - \lambda^2 (I_\rho(x)\phi'(x))' + (EI(x)\phi''(x))'' = 0, \quad 0 < x < 1,$$
$$\phi(0) = \phi'(0) = 0, \quad EI(1)\phi''(1) + \alpha\lambda\phi'(1) = 0,$$
$$(EI\phi'')'(1) - \lambda^2 I_\rho(1)\phi'(1) - \beta\lambda\phi(1) = 0. \tag{2.12}$$

Lemma 2.3. *Let $h_1(x), h_2(x)$ be two linearly independent solutions for the second-order linear homogeneous differential equation*

$$(I_\rho(x)\phi'(x))' - \rho(x)\phi(x) = 0, \tag{2.13}$$

then we have

$$D := h_1(0)h_2'(1) - h_1'(1)h_2(0) \neq 0. \tag{2.14}$$

Lemma 2.4. *If $\alpha + \beta > 0$, then*

$$\mathrm{Re}(\lambda) < 0. \tag{2.15}$$

The proofs of the above two lemmas are just direct verifications (see [13]).

To further simplify (2.12), we expand it to yield

$$\phi^{(4)} + 2\frac{EI'}{EI}\phi''' + \frac{EI''}{EI}\phi'' - \lambda^2\left(\frac{I_\rho}{EI}\phi'' + \frac{I_\rho'}{EI}\phi' - \frac{\rho}{EI}\phi\right) = 0,$$
$$\phi(0) = \phi'(0) = 0, \quad EI(1)\phi''(1) + \alpha\lambda\phi'(1) = 0,$$
$$EI(1)\phi'''(1) + EI'(1)\phi''(1) - \lambda^2 I_\rho(1)\phi'(1) - \beta\lambda\phi(1) = 0. \tag{2.16}$$

Introducing a space-scaling transformation (see [3])

$$\phi(x) := f(z), \quad z := \frac{1}{h}\int_0^x \left(\frac{I_\rho(\zeta)}{EI(\zeta)}\right)^{1/2} \mathrm{d}\zeta, h := \int_0^1 \left(\frac{I_\rho(\zeta)}{EI(\zeta)}\right)^{1/2} \mathrm{d}\zeta, \tag{2.17}$$

then Eq. (2.16) can be rewritten as

$$f^{(4)}(z) + a(z)f'''(z) + b(z)f''(z) + c(z)f'(z) - h^2\lambda^2[f''(z) + d(z)f'(z) - e(z)f(z)] = 0,$$
$$f(0) = f'(0) = 0, \quad b_{21}f''(1) + b_{22}f'(1) + b_{23}\alpha\lambda f'(1) = 0,$$
$$b_{11}f'''(1) + b_{12}f''(1) + b_{13}f'(1) - \lambda^2 b_{14}f'(1) - \beta\lambda f(1) = 0. \tag{2.18}$$

Here

$$a(z) := 6\frac{z_{xx}}{z_x^2} + 2\frac{1}{z_x}\frac{EI'(x)}{EI(x)}, z_x = \frac{1}{h}\left(\frac{I_\rho(x)}{EI(x)}\right)^{1/2}, \tag{2.19}$$

$$b(z) := 3\frac{z_{xx}^2}{z_x^4} + 4\frac{z_{xxx}}{z_x^3} + 6\frac{z_{xx}}{z_x^3}\frac{EI'(x)}{EI(x)} + \frac{1}{z_x^2}\frac{EI''(x)}{EI(x)}, \tag{2.20}$$

$$c(z) := \frac{z_{xxxx}}{z_x^4} + 2\,\frac{z_{xxx}}{z_x^4}\,\frac{EI'(x)}{EI(x)} + \frac{z_{xx}}{z_x^4}\,\frac{EI''(x)}{EI(x)}, \tag{2.21}$$

$$d(z) := \frac{z_{xx}}{z_x^2} + \frac{1}{h^2 z_x^3}\,\frac{I_\rho'(x)}{EI(x)},$$

$$e(z) := \frac{1}{h^2 z_x^4}\,\frac{\rho(x)}{EI(x)}, \tag{2.22}$$

$$b_{11} := z_x^3(1)EI(1), \quad b_{12} := 3z_x(1)z_{xx}(1)EI(1) + z_x^2(1)EI'(1),$$

$$b_{14} := I_\rho(1)z_x(1), \quad b_{13} := z_{xxx}(1)EI(1) + z_{xx}(1)EI'(1),$$

$$b_{21} := z_x^2(1)EI(1), \quad b_{22} := z_{xx}(1)EI(1), b_{23} := z_x(1). \tag{2.23}$$

If we replace λ by $\mu := h\lambda$, then (2.18) changes to

$$f^{(4)}(z) + a(z)f'''(z) + b(z)f''(z) + c(z)f'(z) - \mu^2[f''(z) + d(z)f'(z) - e(z)f(z)] = 0,$$

$$f(0) = 0, \quad f'(0) = 0, \quad b_{21}f''(1) + b_{22}f'(1) + b_{23}\alpha h^{-1}\mu f'(1) = 0,$$

$$b_{11}f'''(1) + b_{12}f''(1) + b_{13}f'(1) - h^{-2}\mu^2 b_{14}f'(1) - \beta h^{-1}\mu f(1) = 0, \tag{2.24}$$

which is equivalent to Eq. (2.16). In summary, we have the following result.

Theorem 2.1. *$\lambda \in \sigma(\mathscr{A})$ if and only if Eq.* (2.24) *has a nonzero solution $f(z)$ for $\mu := h\lambda$. In addition, the function $\phi(x)$ in the corresponding eigenfunction $(\phi, \lambda\phi)$ of $\mathscr{A}$ is given by* (2.17).

3. Asymptotic expressions of eigenfrequencies

In this section, we shall obtain asymptotic expansions for the eigenvalues of $\mathscr{A}$. The main trick is to treat the fundamental solutions of (2.24) first, and then use them to expand the characteristic determinant of $\mathscr{A}$ and obtain the asymptotic eigenfrequency.

To begin, we use a standard technique of Naimark [7] (see also [3]) and divide the complex plane into four sectors

$$\mathscr{S}_k := \left\{ z \in \mathbb{C} \colon \frac{k\pi}{2} \leqslant \arg z \leqslant \frac{(k+1)\pi}{2} \right\}, \quad k = 0, 1, 2, 3 \tag{3.1}$$

and for each $\mathscr{S}_k$, we will pick ω_1 and ω_2 to be the square roots of -1 so that

$$\mathrm{Re}(\rho\omega_1) \leqslant \mathrm{Re}(\rho\omega_2), \quad \forall \rho \in \mathscr{S}_k. \tag{3.2}$$

In particular, we will choose $\omega_1 := e^{i\pi/2}, \omega_2 := e^{i(3/2)\pi}$ in sector $\mathscr{S}_0$ and re-shuffle them in each the remaining sectors so that (3.2) holds. Writing $\mu := \rho\omega_1$ for ρ in each sector $\mathscr{S}_k$, we have the following result on the fundamental solutions of (2.24) from [12, Theorem 3] (see also [1]).

Lemma 3.1. *In each sector $\mathscr{S}_k$, for $\rho \in \mathscr{S}_k$ with $|\rho|$ sufficiently large, the equation*

$$f^{(4)}(z) + a(z)f'''(z) + b(z)f''(z) + c(z)f'(z) + \rho^2[f''(z) + d(z)f'(z) - e(z)f(z)] = 0 \tag{3.3}$$

has four linearly independent fundamental solutions $y_s(z;\rho)\,(s=1,2,3,4)$ *and they possess the following asymptotic expressions* (*for* $j=0,1,2,3$):

$$y_s^{(j)}(z;\rho)=h_s^{(j)}(z)+\mathcal{O}(\rho^{-2}),\quad s=1,2, \tag{3.4}$$

$$y_s^{(j)}(z;\rho)=(\rho\omega_{s-2})^j e^{\rho\omega_{s-2}x}[y_0(z)+\mathcal{O}(\rho^{-1})],\quad s=3,4, \tag{3.5}$$

$$y_0(z):=e^{-(1/2)\int_0^z (a(t)-d(t))\,dt}. \tag{3.6}$$

Here, $h_1(z):=h_1(x(z))$, $h_2(z):=h_2(x(z))$ *are the two linearly independent solutions of* (2.13) *after the transformation* $x(z):=z(x)^{-1}$. *That is, they are two linearly independent solutions of*

$$f''(z)+d(z)f'(z)-e(z)f(z)=0.$$

From (3.4) and (3.5), we can obtain asymptotic expansions for the boundary conditions of system (2.24). For brevity, we shall use the following notation in the sequel:

$$[a]_1:=a+\mathcal{O}(\rho^{-1}).$$

Theorem 3.1. *Denote the boundary conditions of system* (2.7) *respectively by* U_1, U_2, U_3 *and* U_4. *Then, for* $\rho\in\mathcal{S}_0$ *with* $|\rho|$ *sufficiently large, we have the following asymptotic expansions*:

$$U_4(y_s;\rho)=y_s(0;\rho)=\begin{cases} h_s(0)+\mathcal{O}(\rho^{-2}):=[h_s(0)]_1, & s=1,2,\\ 1+\mathcal{O}(\rho^{-1}):=[1]_1, & s=3,4,\end{cases} \tag{3.7}$$

$$U_3(y_s;\rho)=y_s'(0;\rho)=\begin{cases} x_z(0)h_s'(0)+\mathcal{O}(\rho^{-2}):=[x_z(0)h_s'(0)]_1, & s=1,2,\\ \rho\omega_{s-2}(1+\mathcal{O}(\rho^{-1})):=\rho\omega_{s-2}[1]_1, & s=3,4,\end{cases} \tag{3.8}$$

$$\begin{aligned} U_2(y_s;\rho)=&\ y_s''(1;\rho)+\frac{b_{22}}{b_{21}}\,y_s'(1;\rho)+\mathrm{i}\,\frac{b_{23}}{b_{21}}\,\alpha h^{-1}\rho y_s'(1;\rho)\\ =&\begin{cases} \rho\left(\mathrm{i}\,\dfrac{b_{23}}{b_{21}}\,\alpha h^{-1}x_z(1)h_s'(1)+\mathcal{O}(\rho^{-1})\right), & s=1,2,\\ \rho^2 e^{\rho\omega_{s-2}}\left(y_0(1)\omega_{s-2}^2+\mathrm{i}\,\dfrac{b_{23}}{b_{21}}\,\alpha h^{-1}y_0(1)\omega_{s-2}+\mathcal{O}(\rho^{-1})\right), & s=3,4,\end{cases}\\ :=&\begin{cases} \rho\left[\mathrm{i}\,\dfrac{b_{23}}{b_{21}}\,\alpha h^{-1}x_z(1)h_s'(1)\right]_1, & s=1,2,\\ \rho^2 e^{\rho\omega_{s-2}}\left[y_0(1)\omega_{s-2}^2+\mathrm{i}\,\dfrac{b_{23}}{b_{21}}\,\alpha h^{-1}y_0(1)\omega_{s-2}\right]_1, & s=3,4,\end{cases} \end{aligned} \tag{3.9}$$

$$U_1(y_s;\rho)=\ y_s'''(1;\rho)+\frac{b_{12}}{b_{11}}\,y_s''(1;\rho)+\frac{b_{13}}{b_{11}}\,y_s'(1;\rho)+\ \rho^2\,\frac{b_{14}}{h^2b_{11}}\,y_s'(1;\rho)-\mathrm{i}\,\frac{\beta}{hb_{11}}\,\rho y_s(1,\rho)$$

$$= \begin{cases} \rho^2 \left(\dfrac{b_{14}}{b_{11}} h^{-2} x_z(1) h_s'(1) + \mathcal{O}(\rho^{-1}) \right), & s = 1,2, \\ \rho^3 e^{\rho\omega_{s-2}} \left(y_0(1)\omega_{s-2}^3 + \dfrac{b_{14}}{b_{11}} h^{-2} y_0(1)\omega_{s-2} + \mathcal{O}(\rho^{-1}) \right), & s = 3,4, \end{cases}$$

$$:= \begin{cases} \rho^2 [x_z(1) h_s'(1)]_1, & s = 1,2, \\ \rho^3 e^{\rho\omega_{s-2}} [y_0(1)\omega_{s-2}^3 + y_0(1)\omega_{s-2}]_1, & s = 3,4. \end{cases} \tag{3.10}$$

Proof. The proof is just a direct substitution of the fundamental solutions (3.4) and (3.5) into the boundary conditions and makes use of the fact that in (3.10),

$$\frac{b_{14}}{b_{11}} = \frac{I_\rho(1) z_x(1)}{z_x^3(1) EI(1)} = h^2. \qquad \square$$

Since the zeros of the characteristic determinant

$$\Delta(\rho) := \begin{vmatrix} U_4(y_1,\rho) & U_4(y_2,\rho) & U_4(y_3,\rho) & U_4(y_4,\rho) \\ U_3(y_1,\rho) & U_3(y_2,\rho) & U_3(y_3,\rho) & U_3(y_4,\rho) \\ U_2(y_1,\rho) & U_2(y_2,\rho) & U_2(y_3,\rho) & U_2(y_4,\rho) \\ U_1(y_1,\rho) & U_1(y_2,\rho) & U_1(y_3,\rho) & U_1(y_4,\rho) \end{vmatrix} \tag{3.11}$$

are the eigenvalues of (2.12) (see. [7, pp. 13–15]), to estimate the eigenvalues, we substitute (3.7)–(3.10) into $\Delta(\rho)$ and obtain the following asymptotic expansion for them.

Theorem 3.2. *In sector $\mathcal{S}_0$, the characteristic determinant $\Delta(\rho)$ of the characteristic Eq.* (2.24) *has an asymptotic expansion*

$$\Delta(\rho) = -i\rho^5 y_0(1) x_z(1) D\{e^{-i\rho}(1-\alpha\gamma) + e^{i\rho}(1+\alpha\gamma) + \mathcal{O}(\rho^{-1})\}, \tag{3.12}$$

where $\gamma := (I_\rho(1)EI(1))^{-1/2}, D := (h_2'(1)h_1(0) - h_1'(1)h_2(0))$ the non-zero determinant defined in (2.14). *The asymptotic expansion* (3.12) *also holds in the other sectors as well. Furthermore, the boundary problem* (2.24) *is strongly regular in the sense of* [11, p. 259] *if and only if the following condition holds*:

$$1 - \alpha(I_\rho(1)EI(1))^{-1/2} \neq 0 \quad (\textit{i.e. } 1 - \alpha\gamma \neq 0). \tag{3.13}$$

Therefore when (3.13) *holds, the zeros of $\Delta(\rho)$ are simple when their modulus are sufficiently large.*

Proof. In sector $\mathcal{S}_0$, with $\omega_1 := i, \omega_2 := -i$, we conclude that

$$U_1(y_s,\rho) = \rho^3 e^{\rho\omega_{s-2}}[0]_1, \quad s = 3,4, \tag{3.14}$$

$$U_2(y_s,\rho) = \rho^2 e^{\rho\omega_{s-2}} \left[-y_0(1) + (-1)^s \frac{b_{23}}{b_{21}} \alpha h^{-1} y_0(1) \right]_1, \quad s = 3,4. \tag{3.15}$$

Substituting (3.7)–(3.10), (3.14) and (3.15) into the characteristic determinant (3.11), we have

$$\Delta(\rho) = \begin{vmatrix} [h_1(0)]_1 & [h_2(0)]_1 \\ [x_z(0)h_1'(0)]_1 & [x_z(0)h_2'(0)]_1 \\ \rho[i\,\frac{b_{23}}{b_{21}}\,\alpha h^{-1} x_z(1)h_1'(1)]_1 & \rho[i\,\frac{b_{23}}{b_{21}}\,\alpha h^{-1} x_z(1)h_2'(1)]_1 \\ \rho^2[x_z(1)h_1'(1)]_1 & \rho^2[x_z(1)h_2'(1)]_1 \end{vmatrix}$$

$$\begin{matrix} [1]_1 & [1]_1 \\ \mathrm{i}\rho[1]_1 & -\mathrm{i}\rho[1]_1 \\ \rho^2 \mathrm{e}^{\rho\omega_1}[-y_0(1) - \frac{b_{23}}{b_{21}}\alpha h^{-1} y_0(1)]_1 & \rho^2 \mathrm{e}^{\rho\omega_2}[-y_0(1) + \frac{b_{23}}{b_{21}}\alpha h^{-1} y_0(1)]_1 \\ \rho^3 \mathrm{e}^{\rho\omega_1}[0]_1 & \rho^3 \mathrm{e}^{\rho\omega_2}[0]_1 \end{matrix} \Bigg|$$

$$= -\mathrm{i}\rho^5 y_0(1) x_z(1) D \left\{ \mathrm{e}^{\rho\omega_2} \left[1 - \frac{b_{23}}{b_{21}} \alpha h^{-1} \right]_1 + \mathrm{e}^{\rho\omega_1} \left[1 + \frac{b_{23}}{b_{21}} \alpha h^{-1} \right]_1 \right\}.$$

Combining with (2.17) and (2.23), we have

$$\frac{b_{23}}{b_{21}} = \frac{z_x(1)}{z_x^2(1) EI(1)} = h(I_\rho(1) EI(1))^{-1/2} = h\gamma, \tag{3.16}$$

which yields (3.12).

The strong regularity defined in [11, Definition 2.7] together with the simplicity of the zeros with large enough moduli can be verified directly from the fact that $y_0(1), x_z(1) > 0$ and (2.14), (3.13). □

Remark 3.1. In Theorem 3.2, we show that the zeros of $\Delta(\rho)$ are simple provided $|\rho|$ large enough. The theory of ordinary differential equation asserts that the algebraic multiplicity for an non-zero eigenvalue of an ordinary differential operator L with appropriate boundary condition is equal to the multiplicity for it as a zero of the characteristic determinant $\Delta(\rho)$ (see, [5, pp. 92–95]). So we can say that the eigenvalues of $\mathscr{A}$ are geometrically simple provided their moduli are large enough. On the other hand, it can be verified that the eigenvalues of $\mathscr{A}$ with large moduli are also algebraically simple, which can be concluded from the general formula: $m_a \leqslant p \cdot m_g$ (see [6, p. 148]), where p denotes the pole of the resolvent operator and $m_{\mathrm{a}}, m_{\mathrm{g}}$ denote the algebraic and geometric multiplicities, respectively.

Theorem 3.3. *Suppose that condition* (3.13) *is fulfilled, then there exists an integer* $N > 0$ *such that the eigenvalues* λ_k *of problem* (2.12) *have the following asymptotic behavior*:

$$\lambda_k = \frac{1}{h}\left(\tfrac{1}{2}\xi_0 + k\pi \mathrm{i}\right) + \mathcal{O}(k^{-1}), \qquad |k| \geqslant N,\ k \in \mathbb{Z}, \tag{3.17}$$

where

$$h := \int_0^1 \left(\frac{I_\rho(\zeta)}{EI(\zeta)} \right)^{1/2} \mathrm{d}\zeta \quad \textit{and} \quad \xi_0 := \begin{cases} \ln \frac{\alpha\gamma - 1}{\alpha\gamma + 1}, & \alpha\gamma > 1, \\ \ln \frac{1 - \alpha\gamma}{1 + \alpha\gamma} + \pi \mathrm{i}, & \alpha\gamma < 1. \end{cases}$$

Also,

$$\operatorname{Re} \xi_0 = \ln \left| \frac{\alpha\gamma - 1}{\alpha\gamma + 1} \right| < 0 \quad \textit{and} \quad \operatorname{Re} \lambda_k \to \frac{1}{2h} \operatorname{Re} \xi_0 < 0 \quad \textit{as } k \to \infty. \tag{3.18}$$

Proof. Since Eq. (2.12) is equivalent to (2.24), in sector $\mathscr{S}_0$, we see from (3.12) and (2.14) that equation $\Delta(\rho) = 0$ becomes

$$\mathrm{e}^{-\mathrm{i}\rho}(1 - \alpha\gamma) + \mathrm{e}^{\mathrm{i}\rho}(1 + \alpha\gamma) + \mathcal{O}(\rho^{-1}) = 0. \tag{3.19}$$

Solving the equation with lower order terms,

$$\mathrm{e}^{-\mathrm{i}\rho}(1 - \alpha\gamma) + \mathrm{e}^{\mathrm{i}\rho}(1 + \alpha\gamma) = 0$$

will yield solutions

$$\tilde{\mu}_k = \mathrm{i}\rho_k = \tfrac{1}{2}\,\xi_0 + k\pi\mathrm{i}, \quad k = 1, 2, \ldots\,. \tag{3.20}$$

Applying Rouche's theorem to (3.19), then there exists an integer $N > 0$ such that its solutions are expressed by

$$\mu_k = \tfrac{1}{2}\,\xi_0 + k\pi\mathrm{i} + \mathcal{O}(k^{-1}), \quad k \geqslant N, \ k \in \mathbb{N}. \tag{3.21}$$

Note that the eigenvalues of $\mathscr{A}$ are distributed symmetrically with respect to the real axis. So the dual eigenvalues are

$$\mu_k = \tfrac{1}{2}\,\xi_0 - k\pi\mathrm{i} + \mathcal{O}(k^{-1}), \quad k \geqslant N, \ k \in \mathbb{N}. \tag{3.22}$$

Hence, we can conclude from (3.21), (3.22) and $\mu = h\lambda$ that

$$\lambda_k = \frac{1}{h}\,\mu_k = \frac{1}{h}\,\left(\tfrac{1}{2}\,\xi_0 + k\pi\mathrm{i}\right) + \mathcal{O}(k^{-1}), \quad |k| \geqslant N, \ k \in \mathbb{Z}. \tag{3.23}$$

So all eigenvalues of $\mathscr{A}$ are given by (3.23) and the proof is then completed. □

Remark 3.2. Form Theorem 3.3, we know that for $|k| \geqslant N$, the eigenvalues are simple and the asymptotic expressions can be given by (3.23). But we cannot say that the low frequency of $\mathscr{A}$ has $2(N-1)$ number of elements. Indeed, we do not know any exact number of the eigenvalues and the algebraic multiplicity of each eigenvalue in finite domain. Even, so it is not an obstacle for the verification of Riesz basis property because only the asymptotic eigenfrequencies matter.

All the above discussions can be summarized into the following result on the spectrum of $\mathscr{A}$.

Theorem 3.4. *Let $\mathscr{A}$ be defined as in* (2.6) *and* (2.7). *Then each $\lambda \in \sigma(\mathscr{A})$ is an eigenvalue and, when* (3.13) *holds, each λ is algebraically simple when $|\lambda|$ is large enough, and has an asymptotic expression given by* (3.17).

4. Completeness and Riesz basis property for system (2.8)

In this section we will first establish the completeness of the generalized eigenfunctions of $\mathscr{A}$, and then show that they form a Riesz basis. We begin with the following lemma.

Lemma 4.1. *Let $\mathscr{A}$ be defined as in* (2.6) *and* (2.7) *and v_k $(k \in \mathbb{Z})$ be an numeration of all eigenvalues of $\mathscr{A}$, and let $\delta > 0$. Then there exists a constant $M > 0$ such that, for any $\lambda \in \rho(\mathscr{A})$ with $|\lambda - v_k| > \delta$, $k \in \mathbb{Z}$, it holds that*

$$\|R(\lambda, \mathscr{A})\| \leqslant M|\lambda|^2. \tag{4.1}$$

Proof. Let $\lambda \in \rho(\mathscr{A})$ and $(\phi, \psi) \in \mathscr{H}$, we consider the resolvent equation $[\lambda I - \mathscr{A}](f, g) = (\phi, \psi)$, i.e.,

$$\lambda f - g = \phi, \quad \lambda g + C^{-1}(Af + \alpha Dg + \beta Bg) = \psi. \tag{4.2}$$

Simplifying the second equation in (4.2), we have

$$\lambda[\rho g - (I_\rho g')'] + (EIf'')'' = [\rho\psi - (I_\rho \psi')']$$

with boundary conditions $f(0) = f'(0) = 0$, $EIf''|_{x=1} + \alpha g'(1) = 0$ and

$$(EIf'')'|_{x=1} + I_\rho(\psi - \lambda g)'|_{x=1} - \beta g(1) = 0.$$

Thus $g = \lambda f - \phi$ and f satisfies the following equations:

$$\lambda^2[\rho f - (I_\rho f')'] + (EIf'')'' = F(\cdot, \lambda), \tag{4.3}$$

$$f(0) = f'(0) = 0, EIf''|_{x=1} + \alpha\lambda f'(1) = \alpha\phi(1), \tag{4.4}$$

$$(EIf'')'(1) - \lambda^2 I_\rho(1) f'(1) - \beta\lambda f(1) = -v(\lambda), \tag{4.5}$$

where

$$F(x, \lambda) := [\rho(x)(\psi(x) + \lambda\phi(x)) - (I_\rho(x)[\psi'(x) + \lambda\phi'(x)])'],$$

$$v(\lambda) := I_\rho(1)(\psi'(1) + \lambda\phi'(1)) + \beta\phi(1). \tag{4.6}$$

Let $y_j(x, \lambda)$ $(j = 1, 2, 3, 4)$ be the fundamental solutions of the homogenous characteristic equation (2.12). Then any solution $f(x)$ of (4.3)–(4.5) can be expressed by the formula

$$f(x, \lambda) = \int_0^1 G(x, \xi, \lambda)\,(F(\xi, \lambda) + \lambda^2[\rho(\xi)\varphi(\xi) - (I_\rho(\xi)\varphi'(\xi))'] + (EI(\xi)\varphi''(\xi))'')\,\mathrm{d}\xi - \varphi(x), \tag{4.7}$$

where $\varphi(x) := c_1 x^2 + c_2 x^3$ with c_1, c_2 being bound and given by

$$c_1 := \frac{c_{11}\lambda^2 - c_{12}\lambda - 6c_{13}}{c^*}, \quad c_2 := \frac{c_{21}\lambda^2 + c_{22}\lambda + c_{23}}{c^*},$$

$$c_{11} := 3\alpha I_\rho(1)(\phi(1) - \phi'(1)), \quad c_{12} := 2\alpha\beta\phi(1) + I_\rho(1)(6\phi'(1)EI(1) + 3\alpha\psi'(1)),$$

$$c_{13} := \alpha\phi(1)(EI(1) + EI'(1)) + EI(1)(I_\rho(1)\psi'(1) + \beta\phi(1)),$$

$$c_{21} := 2\alpha I_\rho(1)(\phi'(1) - \phi(1)),$$

$$c_{22} := 2I_\rho(1)(EI(1)\phi'(1) + \alpha\psi'(1)) + \alpha\beta\phi(1),$$

$$c_{23} := 2EI(1)(I_\rho(1)\psi'(1) + \beta\phi(1)) + 2\alpha\phi(1)EI'(1),$$

$$c^* := \lambda^2(6EI(1)I_\rho(1) + \alpha\beta) + \lambda(4\beta EI(1) + 6\alpha EI'(1) + 12\alpha EI(1)) + 12EI^2(1)$$

and $G(x, \xi, \lambda)$ is the Green's function given by $G(x, \xi, \lambda) := (1/\Delta(\rho))H(x, \xi, \lambda)$ with

$$H(x, \xi, \rho) := \begin{vmatrix} y_1(x,\lambda) & y_2(x,\lambda) & y_3(x,\lambda) & y_4(x,\lambda) & \eta(x,\xi,\lambda) \\ U_1(y_1) & U_1(y_2) & U_1(y_3) & U_1(y_4) & U_1(\eta) \\ U_2(y_1) & U_2(y_2) & U_2(y_3) & U_2(y_4) & U_2(\eta) \\ U_3(y_1) & U_3(y_2) & U_3(y_3) & U_3(y_4) & U_3(\eta) \\ U_4(y_1) & U_4(y_2) & U_4(y_3) & U_4(y_4) & U_4(\eta) \end{vmatrix}, \tag{4.8}$$

$$\eta(x, \xi, \lambda) :=\tfrac{1}{2}\,\mathrm{sign}(x - \xi)\sum_{j=1}^{4} y_j(x, \lambda) z_j(\xi, \lambda), \tag{4.9}$$

$z_j(x,\lambda) := (W_j(x,\lambda)/W(x,\lambda))$, $W(x,\lambda)$ the Wronskian determinant of $\{y, y_2, y_3, y_4\}$ and $W_j(x,\lambda)$ the cofactor of $y_j(x,\lambda)$ in $W(x,\lambda)$. Substituting the asymptotic expressions (3.4), (3.5) and (3.7)–(3.10) into (4.8) and (4.9), we obtain that for $\lambda\in\rho(\mathscr{A})$ with $|\lambda|$ large enough, there exists a constant M independent of $x,\xi\in[0,1]$ so that

$$\left|\frac{\partial^j}{\partial x^j}H(x,\xi,\lambda)\right| \leqslant M|\lambda|^{5+j}e^{|\lambda|}, \quad j=0,1,2, \tag{4.10}$$

where we have used the relation $\rho\omega_1 = h\lambda$ for all $\lambda\in\mathbb{C}$. Also keeping in mind that $\lambda\in\rho(\mathscr{A})$ with $|\lambda - v_k| \geqslant \delta$, $\forall k\in\mathbb{Z}$, by (3.12) we have

$$\left|\frac{\partial^j}{\partial x^j}G(x,\xi,\lambda)\right| \leqslant M_1|\lambda|^j, \quad j=0,1,2, \tag{4.11}$$

where M_1 is some constant that is independent of $x,\xi\in[0,1]$. These will in turn yield estimates for $f(x)$ and its derivatives, for $j=0,1,2$,

$$\begin{aligned}|f^{(j)}(x)| &\leqslant \int_0^1 \left|\frac{\partial^j}{\partial x^j}G(x,\xi,\lambda)(F(\xi,\lambda)+\lambda^2[\rho(\xi)\varphi(\xi)-(I_\rho(\xi)\varphi'(\xi))'])\right| d\xi + |\varphi^{(j)}(x)|\\ &\leqslant M_1|\lambda^j|\int_0^1 |F(\xi,\lambda)+\lambda^2[\rho(\xi)\varphi(\xi)-(I_\rho(\xi)\varphi'(\xi))']|\,d\xi + |\varphi^{(j)}(x)|.\end{aligned}$$

Eventually, we have the following estimate on the resolvent operator:

$$\begin{aligned}\|(f,g)\|^2 &= \int_0^1 EI(x)|f''(x)|^2\,dx + \int_0^1 \rho(x)|g(x)|^2 + I_\rho(x)|g'(x)|^2\,dx\\ &\leqslant \int_0^1 EI(x)|f''(x)|^2\,dx + |\lambda|^2\int_0^1 [\rho(x)|f(x)|^2 + I_\rho(x)|f'(x)|^2]\,dx + 2\|\phi\|_V\\ &\leqslant M_2^2|\lambda^4|[\|\phi\|_W^2 + \|\psi\|_v^2],\end{aligned} \tag{4.12}$$

where M_2 is some constant independent on $|\lambda|$. So $\|R(\lambda,\mathscr{A})\| \leqslant M_2|\lambda|^2$. $\square$

Theorem 4.1. *Let $\mathscr{A}$ be defined as in* (2.6) *and* (2.7). *If condition* (3.13) *is fulfilled, then the system of the generalized eigenfunctions of $\mathscr{A}$ is complete in Hilbert space $\mathscr{H}$.*

Proof. Let $\sigma(\mathscr{A}) = \{v_n\colon n\in\mathbb{N}\}$ and P_n be the Riesz projection associated with v_n. Denote

$$Sp(\mathscr{A}) = \left\{\sum_{k=1}^N P_k y : y\in\mathscr{H}, \forall N\in\mathbb{N}\right\} \quad \text{and} \quad \mathscr{Q} = \{y\in\mathscr{H}\colon P_k^* y = 0, \forall k\in\mathbb{N}\}.$$

So $\mathscr{H}$ has an orthogonal decomposition $\mathscr{H} = \overline{Sp(\mathscr{A})}\oplus\mathscr{Q}$ and hence the system of generalized eigenfunctions of $\mathscr{A}$ is complete in $\mathscr{H}$ if and only if $\mathscr{Q}=\{0\}$.

Now for any $z\in\mathscr{Q}$, $R(\lambda,\mathscr{A}^*)z$ is an entire function on the complex plane $\mathbb{C}$ valued in $\mathscr{H}$. Since $\|R(\lambda,\mathscr{A}^*)\| = \|R^*(\bar\lambda,\mathscr{A})\| = \|R(\bar\lambda,\mathscr{A})\|$, by Lemma 4.1, $R(\lambda,\mathscr{A}^*)z$ has growth of order at most 2. Hence, $R(\lambda,\mathscr{A}^*)z$ is a polynomial in λ of degree at most two, that is,

$$(\lambda I - \mathscr{A}^*)^{-1}z = d_0 + \lambda d_1 + \lambda^2 d_2 \quad \text{with } d_j\in\mathscr{D}(\mathscr{A}^*), \quad j=0,1,2.$$

Thus,

$$z = (\lambda I - \mathscr{A}^*)(d_0 + \lambda d_1 + \lambda^2 d_2)$$
$$= -\mathscr{A}^* d_0 + \lambda(d_0 - \mathscr{A}^* d_1) + \lambda^2(d_1 - \mathscr{A}^* d_2) + \lambda^3 d_2, \quad \forall \lambda \in \mathbb{C}.$$

Comparing the coefficients of λ^j, we see that $d_0 = d_1 = d_2 = 0$. Therefore $z = 0$, which implies $\mathscr{Q} = \{0\}$. □

Theorem 4.2. *Let X be a separable Hilbert space, and $\mathscr{A}$ be the generator of a C_0 semigroup $T(t)$. Suppose that the following three conditions hold*:

(1) *We can decompose $\sigma(\mathscr{A}) = \sigma_1(\mathscr{A}) \cup \sigma_2(\mathscr{A})$ so that $\sigma_2(\mathscr{A}) = \{\lambda_k\}_{k=1}^{\infty}$ consists of only isolated eigenvalues of finite multiplicity.*

(2) *For $m_a(\lambda_k) := \dim E(\lambda_k, \mathscr{A})X$, where $E(\lambda_k, \mathscr{A})$ is the Riesz projector associated with λ_k, we have*

$$\sup_{k \geqslant 1} m_a(\lambda_k) < \infty. \tag{4.13}$$

(3) *There is a constant α such that*

$$\sup\{\operatorname{Re}\lambda \mid \lambda \in \sigma_1(\mathscr{A})\} \leqslant \alpha \leqslant \inf\{\operatorname{Re}\lambda \mid \lambda \in \sigma_2(\mathscr{A})\} \tag{4.14}$$

and

$$\inf_{n \neq m} |\lambda_n - \lambda_m| > 0. \tag{4.15}$$

Then the following assertions are true:

(i) *There exist two $T(t)$-invariant closed subspaces X_1 and X_2 such that $\sigma(\mathscr{A}|_{X_1}) = \sigma_1(\mathscr{A})$, $\sigma(\mathscr{A}|_{X_2}) = \sigma_2(\mathscr{A})$, and $\{E(\lambda_k, \mathscr{A})X_2\}_{k=1}^{\infty}$ forms a Riesz basis of subspaces for X_2. Furthermore,*

$$X = \overline{X_1 \oplus X_2}.$$

(ii) *If $\sup_{k \geqslant 1} \|E(\lambda_k, \mathscr{A})\| < \infty$, then*

$$D(\mathscr{A}) \subset X_1 \oplus X_2 \subset X. \tag{4.16}$$

(iii) *X has the topological direct sum decomposition $X = X_1 \oplus X_2$ if and only if*

$$\sup_{n \geqslant 1} \left\| \sum_{k=1}^{n} E(\lambda_k, \mathscr{A}) \right\| < \infty. \tag{4.17}$$

Remark 4.1. Theorem 4.2 is a result from [15] which has not been appeared publicly. A brief outline of proof for Theorem 4.2 will be shown in the appendix for the sake of completeness.

Combining Theorems 4.1, 4.2 together with Theorem 3.4, we have the following result.

Theorem 4.3. *Assume that* (3.13) *be fulfilled. System* (2.8) *is a Riesz system* (*in the sense that its generalized eigenfunctions form a Riesz basis in $\mathscr{H}$*) *and hence it satisfies the spectrum determined growth condition.*

Proof. For system (2.8), from Theorems 3.3 and 3.4, we may take $\sigma_2(\mathscr{A}) = \sigma(\mathscr{A})$, $\sigma_1(\mathscr{A}) = \{\infty\}$. Theorem 3.4 shows that conditions (2) and (3) in Theorem 4.2 are true. Finally, Theorem 4.1 implies that $X_1 = \{0\}$. Therefore, the first assertion of Theorem 4.2 says that there is a sequence of generalized eigenfunctions of $\mathscr{A}$ that forms a Riesz basis for $\mathscr{H}$. Since the spectrum determined growth condition is a direct consequence of the existence of a Riesz basis, the proof is completed. □

As a consequence of Theorem 4.3, we have a stability result for system (2.8).

Corollary 4.1. *Let condition* (3.13) *be fulfilled with* $\alpha > 0$ *and* $\beta \geqslant 0$. *Then the system* (2.8) *is exponentially stable.*

Proof. Theorem 4.3 ensures the spectrum-determined growth condition $\omega(\mathscr{A}) = \sup\{\mathrm{Re}\,\lambda, \lambda \in \sigma(\mathscr{A})\}$, Lemma 2.4 says that $\mathrm{Re}\,\lambda < 0$ provided $\lambda \in \sigma(\mathscr{A})$ and Theorem 3.3 shows that imaginary axis is not an asymptote of $\sigma(\mathscr{A})$. Therefore $\sup\{\mathrm{Re}\,\lambda\colon \lambda \in \sigma(\mathscr{A})\} < 0$. □

Remark 4.2. The special case that $\rho(x) = EI(x) \equiv 1$ and $I_\rho(x) \equiv \gamma_1 > 0$ was discussed in [2,8]. In this special case, expression (3.17) then becomes

$$\lambda_k = \frac{1}{\sqrt{\gamma_1}}\left(\frac{1}{2}\,\xi_1 + k\pi \mathrm{i}\right) + \mathcal{O}(k^{-1}), \qquad |k| \geqslant N, \ \ k \in \mathbb{Z} \tag{4.18}$$

with

$$\xi_1 = \begin{cases} \ln \dfrac{\alpha - \sqrt{\gamma_1}}{\alpha + \sqrt{\gamma_1}}, & \alpha > \sqrt{\gamma_1}, \\[2ex] \ln \dfrac{\sqrt{\gamma_1} - \alpha}{\alpha + \sqrt{\gamma_1}} + \pi \mathrm{i}, & \alpha < \sqrt{\gamma_1} \end{cases}$$

and

$$\mathrm{Re}\,\lambda_k \to \frac{1}{2\sqrt{\gamma_1}} \ln \left| \frac{\alpha - \sqrt{\gamma_1}}{\alpha + \sqrt{\gamma_1}} \right| < 0, \qquad k \to \infty.$$

So the closer α to $\alpha^* := \sqrt{\gamma_1}$ the larger the damping rate for system (1.1) which is the conjecture made in [2]. However, when we set the control gain $\alpha = \sqrt{\gamma_1}$, then $\sigma(\mathscr{A})$ may contain at most finitely many eigenvalues. This is because $\mathscr{A}$ has compact resolvent, so there are at most finitely many eigenvalues inside each bounded set and (3.12) tells us that there are no eigenvalues at all when their moduli are large enough.

It should point out that the asymptotic expression of the eigenvalues of $\mathscr{A}$ does not include explicitly the parameter β, which means that the parameter β does not take an essential role for the high frequency part of the spectrum of $\mathscr{A}$. However, it plays an important role for the low frequency part of the spectrum of system (1.1). For instance, $\forall \varepsilon > 0$, we can tune the parameter β appropriately so that the decay rate of the system is less than $v := (1/2h)\xi_0$, where v is the asymptote of the spectrum of $\mathscr{A}$.

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Appendix A. An outline of a proof for Theorem 4.2

Proof of Theorem 4.2. Let X be a separable Hilbert space, and $\mathscr{A}$ be the generator of C_0 semigroup $T(t)$ and satisfy conditions (1–3) of Theorem 4.2. For any $\lambda_k \in \sigma_2(\mathscr{A})$, let m_k denotes its finite algebraical multiplicity and $m_k := m_a(\lambda_k)$. Define

$$Sp_{\sigma_2}(\mathscr{A}) := \left\{ \sum_{k=1}^{m} E(\lambda_k, \mathscr{A})\phi \mid \forall \phi \in X; \forall m \in \mathbb{N} \right\} \tag{A.1}$$

and

$$X_2 := \overline{Sp_{\sigma_2}(\mathscr{A})}; \tag{A.2}$$

then it is easy to see that X_2 is a $T(t)$-invariant closed subspace, and hence it is also $\mathscr{A}$-invariant. Note that (4.13)–(4.15) imply that the family $\{\mathscr{E}^{(k)}(\sigma_2)\}$, where $\mathscr{E}^{(k)}(\sigma_2) := \{e^{\lambda_k t}, te^{\lambda_k t}, \ldots, t^{m_k-1}e^{\lambda_k t}\}$ with $k \in \mathbb{N}$, forms a Riesz basis in space $U := \text{span}\{\mathscr{E}^k(\sigma_2); k \geqslant 1\} \subset L^2[0,T]$ for sufficient large $T > 0$.

Since for any $f \in X$ and $\phi \in Sp_{\sigma_2}(\mathscr{A})$, we have $(T(t)\phi, f) \in U$, so for any $\phi \in X_2$, we also have that $(T(t)\phi, f) \in U$ and

$$(T(t)\phi, f) = \sum_{k=1}^{\infty} \sum_{j=0}^{m_k-1} \frac{t^j}{j!} e^{\lambda_k t} ((\mathscr{A} - \lambda_k)^j E(\lambda_k, \mathscr{A})\phi, f).$$

By the property of Riesz basis of $\{\mathscr{E}^{(k)}(\sigma_2)\}$ in U, there exists constants C_1 and C_2 such that

$$C_1 \sum_{k=1}^{\infty} \sum_{j=0}^{m_k-1} \left| \frac{((\mathscr{A} - \lambda_k)^j E(\lambda_k, \mathscr{A})\phi, f)}{j!} \right|^2 \leqslant \int_0^T |(T(t)\phi, f)|^2 \, dt$$

$$\leqslant C_2 \sum_{k=1}^{\infty} \sum_{j=0}^{m_k-1} \left| \frac{((\mathscr{A} - \lambda_k)^j E(\lambda_k, \mathscr{A})\phi, f)}{j!} \right|^2.$$

Let $T(t)$ satisfies $\|T(t)\| \leqslant Me^{\omega t}$, then we conclude from the left-hand side of above inequalities that

$$C_1 \sum_{k=1}^{\infty} \sum_{j=0}^{m_k-1} \left| \frac{((\mathscr{A} - \lambda_k)^j E(\lambda_k, \mathscr{A})\phi, f)}{j!} \right|^2 \leqslant M^2 \frac{e^{2\omega T} - 1}{2\omega} \|\phi\|^2 \|f\|^2, \tag{A.3}$$

which implies that (by taking $m_k = 1$)

$$C_1 \sum_{k=1}^{\infty} |(E(\lambda_k, \mathscr{A})\phi, f)|^2 \leqslant M^2 \frac{e^{2\omega T} - 1}{2\omega} \|\phi\|^2 \|f\|^2$$

and

$$C_1 \sum_{k=1}^{\infty} \|(E(\lambda_k, \mathscr{A})\phi)\|^2 \leqslant M^2 \frac{e^{2\omega T} - 1}{2\omega} \|\phi\|^2. \tag{A.4}$$

Since $T(t)$ is also a C_0 semigroup on X_2 and its generator is $\mathscr{A}|_{X_2}$ with domain $\mathscr{D} = \mathscr{D}(\mathscr{A}) \cap X_2$, so each $\lambda_k \in \sigma_2(\mathscr{A})$ is an isolated eigenvalue of $\mathscr{A}|_{X_2}$ with finite algebraical multiplicity m_k. Let $\mathscr{A}^\dagger$ and $T^\dagger(t)$ be the adjoint operators of $\mathscr{A}$ and $T(t)$ restricted to X_2, respectively. Note that X_2 endowed with $\|\cdot\|_X$ is a Hilbert space, so $T^\dagger(t)$ is a C_0 semigroup and its generator is $\mathscr{A}^\dagger$. Then each $\bar{\lambda}_k \in \sigma(\mathscr{A}^\dagger)$ is also an isolated eigenvalue of $\mathscr{A}^\dagger$ with finite algebraical multiplicity m_k. Moreover, we have that $E^\dagger(\lambda_k, \mathscr{A}) = E(\bar{\lambda}_k, \mathscr{A}^\dagger)$ and for any $f, g \in X_2$, $(T^\dagger(t)g, f) = (g, T(t)f) \in U$ with

$$(T^\dagger(t)g, f) = \sum_{k=1}^{\infty} \sum_{j=0}^{m_k-1} \frac{t^j}{j!} e^{\bar{\lambda}_k t}((\mathscr{A}^\dagger - \bar{\lambda}_k)^j E(\bar{\lambda}_k, \mathscr{A}^\dagger)g, f).$$

Under the same argument as (A.4), we conclude that

$$C_1 \sum_{k=1}^{\infty} \|E^\dagger(\lambda_k, \mathscr{A})g\|^2 \leqslant M^2 \frac{e^{2\omega T} - 1}{2\omega} \|g\|^2. \tag{A.5}$$

For any $\phi \in Sp_{\sigma_2}(\mathscr{A})$, $\phi = \sum_{k=1}^{N} E(\lambda_k, \mathscr{A})\phi$, we conclude from $E^2(\lambda_k, \mathscr{A}) = E(\lambda_k, \mathscr{A})$ that

$$\begin{aligned}\|\phi\|^2 = (\phi, \phi)d &= \left(\sum_{k=1}^{N} E(\lambda_k, \mathscr{A})\phi, \phi\right) \\ &= \sum_{k=1}^{N} (E(\lambda_k, \mathscr{A})\phi, E^{\dagger}(\lambda_k, \mathscr{A})\phi) \\ &\leqslant \left(\sum_{k=1}^{N} \|E(\lambda_k, \mathscr{A})\phi\|^2\right)^{1/2} \left(\sum_{k=1}^{N} \|E^{\dagger}(\lambda_k, \mathscr{A})\phi\|^2\right)^{1/2} \\ &\leqslant \left(\sum_{k=1}^{N} \|E(\lambda_k, \mathscr{A})\phi\|^2\right)^{1/2} \left(\frac{M^2(\mathrm{e}^{2\omega T} - 1)}{2\omega C_1}\right)^{1/2} \|\phi\|,\end{aligned}$$

i.e.,

$$\|\phi\|^2 \leqslant \frac{M^2(\mathrm{e}^{2\omega T} - 1)}{2\omega C_1} \sum_{k=1}^{m} \|E(\lambda_k, \mathscr{A})\phi\|^2.$$

By the limit process, we obtain

$$\|\phi\|^2 \leqslant \frac{M^2(\mathrm{e}^{2\omega T} - 1)}{2\omega C_1} \sum_{k=1}^{\infty} \|E(\lambda_k, \mathscr{A})\phi\|^2, \quad \forall \phi \in X_2.$$

Combining this inequality and (A.4) yields that $\forall \phi \in X_2$,

$$\frac{2\omega C_1}{M^2(\mathrm{e}^{2\omega T} - 1)} \sum_{k=1}^{\infty} \|E(\lambda_k, \mathscr{A})\phi\|^2 \leqslant \|\phi\|^2 \leqslant \frac{M^2(\mathrm{e}^{2\omega T} - 1)}{2\omega C_1} \sum_{k=1}^{\infty} \|E(\lambda_k, \mathscr{A})\phi\|^2. \tag{A.6}$$

Therefore we conclude from (A.6) that $\{E(\lambda_k, \mathscr{A})X_2\}_{k\geqslant 1}$ is a Riesz basis of subspaces in X_2, $f = \sum_{k=1}^{\infty} E(\lambda_k, \mathscr{A})f$ converges unconditionally in X_2, and

$$\sup_{n\geqslant 1} \left\| \sum_{k=1}^{n} E(\lambda_k, \mathscr{A}) \right\|_{X_2} < \infty, \tag{A.7}$$

where $\|\cdot\|_{X_2}$ denotes the norm in X_2.

Define a subspace $X_1 = \mathscr{Q}(\mathscr{A})$ of X by

$$X_1 = \mathscr{Q}(\mathscr{A}) : = \{g \in X \mid E(\lambda_k, \mathscr{A})g = 0, \ \forall k \geqslant 1\}.$$

Evidently, we have that $X_1 \cap X_2 = \{0\}$ and X_1 is a $T(t)$-invariant closed subspace. In order to prove $X = \overline{X_1 \oplus X_2}$, we only need to show that $X_1 + X_2$ is dense in X. For this we now consider the conjugate $\mathscr{A}^*$ of $\mathscr{A}$ because $T^*(t)$ is also a C_0 semigroup on Hilbert space X with its generator $\mathscr{A}^*$. Note that $\sigma(\mathscr{A}^*) = \{\bar{\lambda} \mid \lambda \in \sigma(\mathscr{A})\} = \overline{\sigma_1(\mathscr{A})} \cup \overline{\sigma_2(\mathscr{A})}$, $\bar{\lambda}_k \in \overline{\sigma_2(\mathscr{A})}$ is an isolated eigenvalue of $\mathscr{A}^*$ with finite algebraical multiplicity m_k, and $E^*(\lambda_k, \mathscr{A}) = E(\bar{\lambda}_k, \mathscr{A}^*)$, then we define

$$X_2^* := \overline{\mathrm{span}\{E^*(\lambda_k, \mathscr{A})X, k \geqslant 1\}}$$

and

$$\mathscr{Q}(\mathscr{A}^*) := \{g \in X \mid E^*(\lambda_k, \mathscr{A})g = 0, \ \forall k \geqslant 1\},$$

so it is easy to see that X_2^* and $\mathcal{Q}(\mathcal{A}^*)$ are $T^*(t)$-invariant closed subspace and $\mathcal{Q}(\mathcal{A}^*)\cap X_2^*=\{0\}$. Furthermore, we have

$$X = \mathcal{Q}(\mathcal{A}^*)\dot{+}X_2 \quad \text{and} \quad X = \mathcal{Q}(\mathcal{A})\dot{+}X_2^*,$$

where sign "$\dot{+}$" denotes the orthogonal sum.

So, for any $h\in X$ such that $h \perp \overline{X_1 + X_2}$, we have $h \perp X_1$ and $h \perp X_2$. Hence, we obtain that $h\in \mathcal{Q}(\mathcal{A}^*)\cap X_2^*$ and $h=0$, which shows $X=\overline{X_1 \oplus X_2}$. Obviously, $\sigma(\mathcal{A}|_{X_2})=\sigma_2(\mathcal{A})$, $\sigma(\mathcal{A}|_{X_1})=\sigma_1(\mathcal{A})$ and the first assertion (i) is concluded.

Secondly, we suppose that $\sup_{k\geqslant 1}\|E(\lambda_k,\mathcal{A})\|_X = M_1 < \infty$. Without loss of generality we assume that $0\in\rho(\mathcal{A})$. Then, for each $f\in X$, we have

$$E(\lambda_k,\mathcal{A})\mathcal{A}^{-1}f = \sum_{j=1}^{m_k}(-1)^{j+1}\frac{(\mathcal{A}-\lambda_k)^{j-1}E(\lambda_k,\mathcal{A})f}{\lambda_k^j}$$

and

$$\|E(\lambda_k,\mathcal{A})\mathcal{A}^{-1}f\|^2 \leqslant \sum_{j=1}^{m_k}\left|\frac{\|(\mathcal{A}-\lambda_k)^{j-1}E(\lambda_k,\mathcal{A})f\|}{(j-1)!}\right|^2\sum_{j=1}^{m_k}\left|\frac{(j-1)!}{\lambda_k^j}\right|^2.$$

Note that we conclude from (A.3) and (4.13) respectively that

$$C_1\sum_{j=0}^{m_k-1}\left|\frac{\|(\mathcal{A}-\lambda_k)^{j}E(\lambda_k,\mathcal{A})f\|}{j!}\right|^2 \leqslant M^2\frac{e^{2\omega T}-1}{2\omega}\|E(\lambda_k,\mathcal{A})f\|^2$$

and

$$\sum_{j=1}^{m_k}\left|\frac{(j-1)!}{\lambda_k^j}\right|^2 \leqslant 2\frac{(N_a!)^2}{|\lambda_k|^2}, \quad \text{for } |\lambda_k|>2,$$

where $N_a := \sup_{k\geqslant 1} m_k < \infty$. Hence we obtain that

$$\begin{aligned}\|E(\lambda_k,\mathcal{A})\mathcal{A}^{-1}f\|^2 &\leqslant \frac{M^2(e^{2\omega T}-1)}{\omega C_1}\frac{(N_a!)^2}{|\lambda_k|^2}\|E(\lambda_k,\mathcal{A})f\|^2\\ &\leqslant \frac{M^2(e^{2\omega T}-1)}{\omega C_1}\frac{(N_a!)^2}{|\lambda_k|^2}M_1^2\|f\|^2\end{aligned}$$

and by condition (4.15), the series $\sum_{k=1}^{\infty}\|E(\lambda_k,\mathcal{A})\mathcal{A}^{-1}f\|^2$ is unconditionally convergent in X. Furthermore, we conclude from the first assertion (i) that $\sum_{k=1}^{\infty}E(\lambda_k,\mathcal{A})\mathcal{A}^{-1}f$ is an element of X_2. Setting

$$g=\mathcal{A}^{-1}f-\sum_{k=1}^{\infty}E(\lambda_k,\mathcal{A})\mathcal{A}^{-1}f,$$

we have $g\in\mathcal{Q}(\mathcal{A})$. So $\mathcal{D}(\mathcal{A})\subset X_1\oplus X_2$ and the second assertion (ii) is followed.

Finally, we suppose that X has a direct sum decomposition $X=X_1\oplus X_2$. Define projection $\mathcal{P}$ from X onto X_2 along X_1, then $\mathcal{P}$ is a bounded operator on X. For each $f\in X$, we have $\mathcal{P}f\in X_2$ and

$$\left\|\sum_{k=1}^{n}E(\lambda_k,\mathcal{A})\mathcal{P}f\right\| \leqslant \left\|\sum_{k=1}^{n}E(\lambda_k,\mathcal{A})\right\|_{X_2}\|\mathcal{P}f\| \leqslant \left\|\sum_{k=1}^{n}E(\lambda_k,\mathcal{A})\right\|_{X_2}\|\mathcal{P}\|\|f\|.$$

So, together with (A.7) we have

$$\sup_{n\geqslant 1}\left\|\sum_{k=1}^{n} E(\lambda_k,\mathscr{A})\right\|_X < \infty. \tag{A.8}$$

Conversely, if (A.8) holds, then (A.4) and (A.8) imply that for any $n\in\mathbb{N}, f\in X$,

$$\sum_{k=1}^{n}\|E(\lambda_k,\mathscr{A})f\|^2 \leqslant \frac{(\mathrm{e}^{2\omega T}-1)}{2M^{-2}C_1\omega}\left\|\sum_{k=1}^{n} E(\lambda_k,\mathscr{A})f\right\|^2$$

$$\leqslant \frac{(\mathrm{e}^{2\omega T}-1)}{2M^{-2}C_1\omega}\left\|\sum_{k=1}^{n} E(\lambda_k,\mathscr{A})\right\|_X^2 \|f\|^2.$$

Therefore,

$$\sum_{k=1}^{\infty}\|E(\lambda_k,\mathscr{A})f\|^2 < \infty,$$

which shows that for any $f\in X$, the series $\sum_{k=1}^{\infty} E(\lambda_k,\mathscr{A})f$ converges unconditionally in X_2. Taking

$$f_1 = f - \sum_{k=1}^{\infty} E(\lambda_k,\mathscr{A})f,$$

then, $f_1\in\mathscr{Q}(\mathscr{A}) = X_1$. Hence $X = X_1\oplus X_2$ and the proof is completed. □

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